Supplementary material to:

On the economic theory of crop rotations: value of the crop rotation effects and implications on acreage choice modeling

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Technical Appendices

Appendix A. Quadratic perturbation

Given our interpretation of $C(\mathbf{s})$, defined as

(A1) $C(\mathbf{s}) = \mathbf{h}'\mathbf{s} + 1/2 \times \mathbf{s}'\mathbf{H}\mathbf{s}$ where $\mathbf{H} \in \mathbb{R}^N \times \mathbb{R}^N$ is positive definite, $\mathbf{h} \in \mathbb{R}^N$ and $\mathbf{s} \in \mathbb{R}_+^N$,

the assumption stating its strict convexity in **s** is debatable. Cropping practices vary much more across crops than they vary across preceding crops for a given crop. Hence, it seems more sensible to define the acreage management cost as a function of the crop acreage vector $\mathbf{a} \equiv \mathbf{A}\mathbf{s}$, *i.e.* as

(A2) $C^a(\mathbf{a}) \equiv \mathbf{h}'\mathbf{a} + 1/2 \times \mathbf{a}'\mathbf{G}\mathbf{a}$ where $\mathbf{G} \in \mathbb{R}^K \times \mathbb{R}^K$ is positive definite, $\mathbf{g} \in \mathbb{R}^K$ and $\mathbf{a} \in \mathbb{R}_+^K$.

Of course $C^a(\mathbf{A}\mathbf{s})$ is strictly convex in $\mathbf{A}\mathbf{s}$ and convex in **s**, but it is not strictly convex in **s**. Our approach consists in defining $C(\mathbf{s})$ as a strictly convex perturbed version of the "only" convex $C^a(\mathbf{A}\mathbf{s})$.

The perturbation technique is a mathematical device which is often used in operation research for obtaining a well-behaved objective function smoothly approximating the objective function of interest. Linear or quadratic programming problems are usually perturbed by quadratic terms. In our case, $C(\mathbf{s}_t)$ can be defined as

(A3) $C(\mathbf{s}_t) \equiv C^a(\mathbf{A}\mathbf{s}_t) + \rho^{-1}\mathbf{s}_t'\mathbf{s}_t$ where $\rho > 0$

by using the perturbation term $\rho^{-1}\mathbf{s}_t'\mathbf{s}_t$. In this case we have

(A4) $\mathbf{H} \equiv \mathbf{A}'\mathbf{G}\mathbf{A} + \rho^{-1}\mathbf{I} \otimes \mathbf{I}$

and it is easily seen that **H** is positive definite as long as $\rho > 0$. This is so even if $\mathbf{G} = \mathbf{0}$, *i.e.* where the original version problem (S) is a linear programming problem. To choose the perturbation parameter ρ as large as possible ensures that the computed solutions are reliable approximations to solutions of interest.

The perturbation term $\rho^{-1}\mathbf{s}_t'\mathbf{s}_t$ is bounded as long as $\mathbf{s}_t$ is confined in a compact set. The feasible sets of the problems considered in the article are all compact. In that case, we know that C uniformly converges to C^s in $\rho \to +\infty$ on the considered compact set. This implies that the solutions obtained with C converge in $\rho \to +\infty$ to solutions of the corresponding problems defined with C^s. This ensures that the solutions to the perturbed versions of the problems of interest are $\varepsilon -$solutions to the problem of interest, *i.e.* that the solutions to the perturbed version of the problem of interest are reliable approximate solutions to the problem of interest. Of course, to choose of very large levels of ρ is innocuous in a theoretical analysis but this can generate ill-conditioning issues in practice.

Appendix B. Static problem

This Appendix provides results related to the characterization of the static problem (S). We first recall the definitions of a polyhedron, of a polyhedral partition and of piecewise linear and piecewise quadratic functions which will be used later.

Definition C1. *Polyhedron*

$\mathcal{P} \subseteq \mathbb{R}^N$ is a polyhedron of $\mathbb{R}^N$ if $\mathcal{P} \equiv \{\boldsymbol{\pi} \in \mathbb{R}^N : \mathbf{C}\boldsymbol{\pi} \leq \mathbf{c}\}$ for some $(\mathbf{c}, \mathbf{C}) \in \mathbb{R}^C \times \mathbb{R}^{C \times N}$.

Definition C2. *Polyhedral partition*. $\{\mathcal{P}_j : j \in \mathcal{J}\}$ is a polyhedral partition of the polyhedron $\mathcal{P} \subseteq \mathbb{R}^N$ if (*a*) $\cup_{j \in \mathcal{J}} \mathcal{P}_j = \mathcal{P}$, (*b*) $\mathcal{P}_j$ is a polyhedron such that $\mathcal{P}_j \subseteq \mathcal{P}$ for any $j \in \mathcal{J}$ and (*c*) $\operatorname{int}\mathcal{P}_j \cap \operatorname{int}\mathcal{P}_i = \varnothing$ for any $(i, j) \in \mathcal{J} \times \mathcal{J}$.

Definition C3. *Piecewise affine and piecewise quadratic*

(*i*) *Piecewise affine*. $\mathbf{f} : \mathcal{P} \to \mathbb{R}^M$ is piecewise affine $\boldsymbol{\pi}$ on $\mathcal{P}$ if there exists a polyhedral partition $\{\mathcal{P}_j : j \in \mathcal{J}\}$ of the polyhedron $\mathcal{P}$ such that $j \in \mathcal{J}$ implies that $\mathbf{f}(\boldsymbol{\pi}) = \mathbf{b}_j + \mathbf{B}_j \boldsymbol{\pi}$ for some $(\mathbf{b}_j, \mathbf{B}_j) \in \mathbb{R}^M \times \mathbb{R}^{M \times N}$. (*ii*) *Piecewise quadratic*. $f : \mathcal{P} \to \mathbb{R}$ is piecewise quadratic in $\boldsymbol{\pi}$ on the polyhedron $\mathcal{P}$ if there exists a polyhedral partition $\{\mathcal{P}_j : j \in \mathcal{J}\}$ of $\mathcal{P}$ such that $j \in \mathcal{J}$ implies that $\mathbf{f}(\boldsymbol{\pi}) = h_j + \boldsymbol{\pi}'\mathbf{h}_j + 1/2 \times \boldsymbol{\pi}'\mathbf{H}_j \boldsymbol{\pi}$ for some $(h_j, \mathbf{h}_j, \mathbf{H}_j) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N \times N}$.

Proposition 1 in the main text is defined as Proposition B1 in this Appendix.

Proposition B1. *Static problem*

Let consider problem (S): $\max_{\mathbf{s}\geq\mathbf{0}} \Pi(\mathbf{s};\boldsymbol{\pi})$ where $\Pi(\mathbf{s};\boldsymbol{\pi}) \equiv \mathbf{s}'\boldsymbol{\pi} - C(\mathbf{s})$ and $(\mathbf{s},\boldsymbol{\pi}) \in \mathbb{R}_+^N \times \mathbb{R}^N$.

Let assume that $C:\mathbb{R}_+^N \to \mathbb{R}$ is quadratic and strictly convex in **s** on $\mathbb{R}_+^N$.

(*i*) The solution in **s** to problem (S) is unique and defines the function $\mathbf{s}^s:\mathbb{R}^N \to \mathbb{R}_+^N$ by

$\mathbf{s}^s(\boldsymbol{\pi}) \equiv \arg\max_{\mathbf{s}\geq\mathbf{0}} \Pi(\mathbf{s};\boldsymbol{\pi})$. $\mathbf{s}^s$ is continuous in $\boldsymbol{\pi}$ on $\mathbb{R}^N$.

(*ii*) The solution to problem (S) defines the value function $\Pi^s:\mathbb{R}^N \to \mathbb{R}$ by

$\Pi^s(\boldsymbol{\pi}) \equiv \max_{\mathbf{s}\geq\mathbf{0}} \Pi(\mathbf{s};\boldsymbol{\pi})$. Π^s is convex in $\boldsymbol{\pi}$

(*iii*) $\mathbf{s}^s$ is piecewise affine and Π^s is piecewise quadratic in $\boldsymbol{\pi}$ on $\mathbb{R}^N$.

(*iv*) Π^s is continuously differentiable on $\mathbb{R}^N$ with $\frac{\partial}{\partial\boldsymbol{\pi}}\Pi^s(\boldsymbol{\pi}) = \mathbf{s}^s(\boldsymbol{\pi})$.

(*v*) s^s_{mk} is strictly increasing in π_{mk} at $\boldsymbol{\pi}$ if $s^s_{mk}(\boldsymbol{\pi}) > 0$. If s^s_{mk} is constant in π_{mk} at $\boldsymbol{\pi}$ then

$s^s_{mk}(\boldsymbol{\pi}) = 0$.

Let define the functions $d^s_m:\mathbb{R}^N \to \mathbb{R}_+$ by $d^s_m \equiv \boldsymbol{\iota}'\mathbf{s}^s_{(m)}$ for $m \in \mathcal{K}$ and $a^s_k:\mathbb{R}^N \to \mathbb{R}_+$ by

$a^s_k \equiv \boldsymbol{\iota}'\mathbf{s}^s_k$ for $k \in \mathcal{K}$. Let $\boldsymbol{\mu} \in \mathbb{R}^K$.

(*vi*) Let $k \in \mathcal{K}$. a^s_k is strictly increasing in μ_k at $\boldsymbol{\pi} + \boldsymbol{\mu}\otimes\boldsymbol{\iota}$ if $a^s_k(\boldsymbol{\pi} + \boldsymbol{\mu}\otimes\boldsymbol{\iota}) > 0$. If a^s_k is

constant in μ_k at $\boldsymbol{\pi} + \boldsymbol{\mu}\otimes\boldsymbol{\iota}$ then $a^s_k(\boldsymbol{\pi} + \boldsymbol{\mu}\otimes\boldsymbol{\iota}) = 0$.

(*vii*) Let $m \in \mathcal{K}$. d^s_m is strictly decreasing in μ_m at $\boldsymbol{\pi} - \boldsymbol{\iota}\otimes\boldsymbol{\mu}$ if $d^s_m(\boldsymbol{\pi} - \boldsymbol{\iota}\otimes\boldsymbol{\mu}) > 0$. If d^s_m is

constant in μ_m at $\boldsymbol{\pi} - \boldsymbol{\iota}\otimes\boldsymbol{\mu}$ then $d^s_m(\boldsymbol{\pi} - \boldsymbol{\iota}\otimes\boldsymbol{\mu}) = 0$.

Proof. The objective function of problem (S) is quadratic and strictly concave. The feasible set of problem (S) is polyhedral and has a non-empty interior. Most of the results collected in Propositions B1 are either well-known or demonstrated in Bemporad *et al* (2002) and Lau and Womersley (2001).

The only results to be demonstrated are the continuous differentiability of Π^s in $\boldsymbol{\pi}$ on $\mathbb{R}^N$ and results (*v*)-(*vii*). These last results are proven later. These results are given in Proposition C1 because they are given in Proposition 1 in the main text.

The continuous differentiability of Π^s in $\boldsymbol{\pi}$ on $\mathbb{R}^N$ is obtained by applying a result due to Jittorntrum (1984) (see also Theorems 4.1 and 7.3 in Fiacco and Kyparisis, 1985). This result implies that the value function of a constrained strictly concave parametric maximization problem in $\mathbf{s}$ with parametric convex constraints is continuously differentiable in its parameters at the parameter values if the Linear Independence Constraint Qualification (LICQ) condition holds and if the considered problem is sufficiently smooth. The LICQ condition holds at an optimum if the gradients in $\mathbf{s}$ of the constraint functions of the active constraints are linearly independent. In the case of problem (S), the only considered constraints are the non-negativity constraints $\mathbf{s} \geq \mathbf{0}$. The gradient is $\mathbf{s}$ of the corresponding active constraint functions is an identity matrix. As a result the LICQ condition holds at any solution to problem (S) and Π^s is continuously differentiable in $\boldsymbol{\pi}$ on $\mathbb{R}^N$. It is then easily shown that $\frac{\partial}{\partial \boldsymbol{\pi}} \Pi^s(\boldsymbol{\pi}) = \mathbf{s}^s(\boldsymbol{\pi})$.

Results (*v*)-(*vii*) are parts of Proposition B3 and their proofs are given in that of Proposition B3. It relies on the specific structure of problem (S), as the other results collected in Proposition B2.

QED.

Result (*iii*) implies that there exists a polyhedral partition $\{\mathcal{P}_j : j \in \mathcal{J}\}$ of $\mathcal{P}$ such that there exists $(\mathbf{b}_j, \mathbf{B}_j) \in \mathbb{R}^M \times \mathbb{R}^{M \times N}$ such that:

(B1) $\mathbf{s}^s(\boldsymbol{\pi}) = \mathbf{b}_j + \mathbf{B}_j \boldsymbol{\pi}$ and $\Pi^s(\boldsymbol{\pi}) = \boldsymbol{\pi}'\mathbf{b}_j + \boldsymbol{\pi}'\mathbf{B}_j\boldsymbol{\pi} + 1/2 \times (\mathbf{b}_j + \mathbf{B}_j\boldsymbol{\pi})'\mathbf{H}(\mathbf{b}_j + \mathbf{B}_j\boldsymbol{\pi})$

if $\boldsymbol{\pi} \in \mathcal{P}_j$ for any $j \in \mathcal{J}$. Π^s is twice continuously differentiable almost everywhere in $\boldsymbol{\pi}$ on $\mathcal{P}$. Π^s is twice continuously differentiable in $\boldsymbol{\pi}$ on $\text{int}\mathcal{P}_j$ for any $j \in \mathcal{J}$. Π^s is C^1 $\boldsymbol{\pi}$ on $\mathcal{P}$ but it is not twice continuously differentiable in $\boldsymbol{\pi}$ on the set collecting the separating frontiers of the polyhedral partition $\{\mathcal{P}_j : j \in \mathcal{J}\}$. This last set has null Lebesgue measure.

Results (*v*)-(*vii*) are parts of Proposition C3 and their proofs are given in that of Proposition B3. These results rely on the specific structure of Problem (S) which is investigated in more depth thanks to the results provided in Proposition B2. These results characterize the functional forms of $\mathbf{s}^s$ and of Π^s. They rely on the following definition the regime of **s**.

Definition B1. *Regimes*

The regime of $\mathbf{s}$, denoted $r(\mathbf{s})$, is characterized by the subset of $\mathcal{N} \equiv \mathcal{K} \times \mathcal{K}$, $\mathcal{N}_{r(\mathbf{s})}$, containing the pairs (m,k) such that $s_{mk} > 0$.

Proposition B2. *Functional forms and further properties of* Π^s *and of* $\mathbf{s}^s$

Let assume that the regimes of the solutions in **s** to problem (S), *i.e*. the regimes of the terms $\mathbf{s}^s(\boldsymbol{\pi})$ with $\boldsymbol{\pi} \in \mathcal{P}$ denoted by $r^s(\boldsymbol{\pi})$, can be characterized by *R* regimes defining the regime set $\mathcal{R} \equiv \{1,...,R\}$.

(*i*) Let define the Lagrangian problem associated to problem (S) as:

(LS) $\quad \min_{\boldsymbol{\lambda} \geq \mathbf{0}} \max_{\mathbf{s}} \{\Pi(\mathbf{s},\boldsymbol{\pi}) + \mathbf{s}'\boldsymbol{\lambda}\}$

where $\boldsymbol{\pi} \in \mathbb{R}^N$ and $\boldsymbol{\lambda}$ is the LM vector associated to the non-negativity constraint $\mathbf{s} \geq \mathbf{0}$. The corresponding Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{cases} \boldsymbol{\pi} - \mathbf{h} + \boldsymbol{\lambda} - \mathbf{Hs} = \mathbf{0} \\ \boldsymbol{\lambda}'\mathbf{s} = 0 \text{ with } \mathbf{s} \geq \mathbf{0} \text{ and } \boldsymbol{\lambda} \geq \mathbf{0} \end{cases}$$

characterize the unique solution in **s** and $\boldsymbol{\lambda}$ to problem (LS), $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$.

Let define $\mathbf{P}_r$ as a the $\dim \mathcal{N}_r \times N$ matrix selecting the subvector of **s** or of $\boldsymbol{\pi}$ corresponding to the subset of indice pairs $\mathcal{N}_r$. Let define $\mathbf{Z}_r$ as a the $(N - \dim \mathcal{N}_r) \times N$ matrix selecting a the subvector of **s** or of $\boldsymbol{\pi}$ corresponding to the subset of indice pairs $\mathcal{N} \setminus \mathcal{N}_r$.

(*ii*) If $r^s(\boldsymbol{\pi}) = r$ then:

$$\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi}) = \mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}) \text{ and } \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi}) = \mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi} - \mathbf{h}),$$

$$\mathbf{s}^s(\boldsymbol{\pi}) = \mathbf{P}_r'\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}) \text{ and } \boldsymbol{\lambda}^s(\boldsymbol{\pi}) = \mathbf{Z}_r'\mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi} - \mathbf{h}),$$

where $\mathbf{G}_r \equiv (\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}$ and $\mathbf{L}_r \equiv \mathbf{H}\mathbf{P}_r'(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}\mathbf{P}_r - \mathbf{I}_N$, and:

$$\Pi^s(\boldsymbol{\pi}) = 1/2 \times (\boldsymbol{\pi} - \mathbf{h})'\mathbf{P}_r'\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}).$$

(*iii*) The regime of the solution $\mathbf{s}^s(\boldsymbol{\pi})$, $r^s(\boldsymbol{\pi})$, is characterized by:

$$r^s(\boldsymbol{\pi}) = r \quad \Leftrightarrow \quad \begin{cases} \mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi}) > \mathbf{0} \;\Leftrightarrow\; \mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}) > \mathbf{0} \\ \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi}) \geq \mathbf{0} \;\Leftrightarrow\; \mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi} - \mathbf{h}) \geq \mathbf{0}. \end{cases}$$

(*iv*) The functions $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$ are piecewise affine and the function $\Pi^s(\boldsymbol{\pi})$ is piecewise quadratic in $\boldsymbol{\pi}$ on $\mathcal{P} \subseteq \mathbb{R}^N$ according to the polyhedral partition $\{\mathcal{P}_r : r \in \mathcal{R}\}$ of $\mathcal{P}$ defined by $\mathcal{P}_r \equiv \{\boldsymbol{\pi} \in \mathcal{P} : \mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}) \geq \mathbf{0} \text{ and } \mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi} - \mathbf{h}) \geq \mathbf{0}\}$.

(*v*) If $\mathbf{s}^s(\boldsymbol{\pi}) \neq \mathbf{0}$, $\Pi^s(\boldsymbol{\pi})$ is strictly convex in $\mathbf{P}_r\boldsymbol{\pi}$ on $\mathcal{P}_r$ with:

$$\tfrac{\partial}{\partial(\mathbf{P}_r\boldsymbol{\pi})}\Pi^s(\boldsymbol{\pi}) = \mathbf{G}_r\mathbf{P}_r\boldsymbol{\pi} \text{ for } \boldsymbol{\pi} \in \mathcal{P}_r, \text{ and } \tfrac{\partial^2}{\partial(\mathbf{P}_r\boldsymbol{\pi})\partial(\mathbf{P}_r\boldsymbol{\pi})'}\Pi^s(\boldsymbol{\pi}) = \mathbf{G}_r \text{ for } \boldsymbol{\pi} \in \operatorname{int}\mathcal{P}_r.$$

Proof. Results (*i*) and (*ii*) are standard. The unicity of $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$ for any $\boldsymbol{\pi} \in \mathbb{R}^N$ follows from the strict concavity Π in **s**, the linearity the involved constraints and the linear independence of the gradients in $\mathbf{s}$ of their corresponding constraints. The definitions of $\mathbf{P}_r$ and $\mathbf{Z}_r$ ensures the equivalence of:

$$\boldsymbol{\pi}-\mathbf{h}+\boldsymbol{\lambda}^s(\boldsymbol{\pi})-\mathbf{H}\mathbf{s}^s(\boldsymbol{\pi})=\mathbf{0}$$

and of:

$$\begin{bmatrix}\mathbf{P}_r(\boldsymbol{\pi}-\mathbf{h})\\ \mathbf{Z}_r(\boldsymbol{\pi}-\mathbf{h})\end{bmatrix}+\begin{bmatrix}\mathbf{P}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi})\\ \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi})\end{bmatrix}-\begin{bmatrix}\mathbf{P}_r\mathbf{H}\mathbf{P}_r' & \mathbf{P}_r\mathbf{H}\mathbf{Z}_r'\\ \mathbf{Z}_r\mathbf{H}\mathbf{P}_r' & \mathbf{Z}_r\mathbf{H}\mathbf{Z}_r'\end{bmatrix}\begin{bmatrix}\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi})\\ \mathbf{Z}_r\mathbf{s}^s(\boldsymbol{\pi})\end{bmatrix}=\mathbf{0}.$$

With $r^s(\boldsymbol{\pi})=r$, the definitions of $\mathbf{P}_r$ and $\mathbf{Z}_r$ and the complementarity condition, *i.e.* $\boldsymbol{\lambda}^s(\boldsymbol{\pi})'\mathbf{s}^s(\boldsymbol{\pi})=0$, $\mathbf{s}^s(\boldsymbol{\pi})\geq\mathbf{0}$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})\geq\mathbf{0}$, provide:

$$\begin{bmatrix}\mathbf{P}_r\boldsymbol{\pi}\\ \mathbf{Z}_r\boldsymbol{\pi}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi})\end{bmatrix}-\begin{bmatrix}\mathbf{P}_r\mathbf{Q}\mathbf{P}_r' & \mathbf{P}_r\mathbf{H}\mathbf{Z}_r'\\ \mathbf{Z}_r\mathbf{Q}\mathbf{P}_r' & \mathbf{Z}_r\mathbf{H}\mathbf{Z}_r'\end{bmatrix}\begin{bmatrix}\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi})\\ \mathbf{0}\end{bmatrix}=\mathbf{0}.$$

Some manipulations yield:

$$\begin{cases}\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi})=(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}\mathbf{P}_r(\boldsymbol{\pi}-\mathbf{h})=\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi}-\mathbf{h})\\ \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi})=\mathbf{Z}_r\mathbf{H}\mathbf{P}_r'(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}\mathbf{P}_r(\boldsymbol{\pi}-\mathbf{h})-\mathbf{Z}_r(\boldsymbol{\pi}-\mathbf{h})=\mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi}-\mathbf{h})\end{cases}$$

the second equalities using $\mathbf{G}_r\equiv(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}$ and $\mathbf{L}_r\equiv\mathbf{H}\mathbf{P}_r'(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')^{-1}\mathbf{P}_r-\mathbf{I}_N$. The matrix $\mathbf{P}_r\mathbf{H}\mathbf{P}_r'$ is positive definite because $\mathbf{H}$ is definite positive and $\mathbf{P}_r$ has full row rank. This in turn ensures that $\mathbf{P}_r\mathbf{H}\mathbf{P}_r'$ is invertible and that its inverse, $\mathbf{G}_r$, is also positive definite. The formula of $\Pi^s(\boldsymbol{\pi})$ is then obtained by substitution with:

$$\Pi^s(\boldsymbol{\pi})=\Pi(\mathbf{s}^s(\boldsymbol{\pi});\boldsymbol{\pi})=(\boldsymbol{\pi}-\mathbf{h})'\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi})-1/2\times\mathbf{s}^s(\boldsymbol{\pi})'\mathbf{P}_r'(\mathbf{P}_r\mathbf{H}\mathbf{P}_r')\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi}).$$

Result (*iii*) follows from the definition of the regimes and from the KKT conditions uniquely characterizing $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$ for any $\boldsymbol{\pi}\in\mathcal{P}\subseteq\mathbb{R}^N$. If $r^s(\boldsymbol{\pi})=r$ then the KKT conditions necessarily provide:

$$\begin{cases}\mathbf{P}_r\mathbf{s}^s(\boldsymbol{\pi})=\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi}-\mathbf{h})>\mathbf{0}\\ \mathbf{Z}_r\boldsymbol{\lambda}^s(\boldsymbol{\pi})=\mathbf{Z}_r\mathbf{L}_r(\boldsymbol{\pi}-\mathbf{h})\geq\mathbf{0}.\end{cases}$$

This condition set must also hold if the KKT conditions imply that $r^s(\boldsymbol{\pi}) = r$. This implies that his condition set is equivalent to $r^s(\boldsymbol{\pi}) = r$, provided that the KKT must hold for $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$ to be optimal.

Result (*iv*) follows from the assumption that the regime set $\mathcal{R}$ exhausts all possible regimes and from results (*i*)-(*iii*). The definition of $\mathcal{P}_r$ shows that it is a polyhedron in $\mathbb{R}^N$. Result (*v*) follows from results (*ii*) and (*iv*). It then suffices to note that $\mathbf{G}_r$ is positive definite and that $\mathbf{P}_r$ has full row rank.

QED.

The inequality $\mathbf{Z}_r \boldsymbol{\lambda}^s(\boldsymbol{\pi}) \geq \mathbf{0}$ being equivalent to $\mathbf{Z}_r \boldsymbol{\pi} \leq (\mathbf{Z}_r \mathbf{H}\mathbf{P}_r' \mathbf{G}_r)\mathbf{P}_r \boldsymbol{\pi} + \mathbf{L}_r \mathbf{h}$, the term

(CB2) $\mathbf{z}_r^s(\mathbf{P}_r \boldsymbol{\pi}) \equiv (\mathbf{Z}_r \mathbf{H}\mathbf{P}_r' \mathbf{G}_r)(\mathbf{P}_r \boldsymbol{\pi}) + \mathbf{L}_r \mathbf{h}$

can be interpreted as the reservation gross margin vector of the crop sequences $\mathcal{N} \setminus \mathcal{N}_r$ in regime *r*. This follows from the fact that $\mathbf{Z}_r \mathbf{p} \leq \mathbf{z}_r^s(\mathbf{P}_r \boldsymbol{\pi})$ and $\mathbf{P}_r \mathbf{p} = \mathbf{P}_r \boldsymbol{\pi}$ imply $\mathbf{s}^s(\mathbf{p}) = \mathbf{s}^s(\boldsymbol{\pi})$ and, in particular, $\mathbf{Z}_r \mathbf{s}^s(\mathbf{p}) = \mathbf{Z}_r \mathbf{s}^s(\boldsymbol{\pi}) = \mathbf{0}$. Such reservation “prices” play an important role in the econometrics of the demand systems with corner solutions.

Proposition B3 provides results related to the sensitivity analysis of $\mathbf{s}^s$ with respect to $\boldsymbol{\pi}$.

Proposition B3. *Kinks of* $\mathbf{s}^s(\boldsymbol{\pi})$, *of* $\mathbf{D}\mathbf{s}^s(\boldsymbol{\pi})$ *and of* $\mathbf{A}\mathbf{s}^s(\boldsymbol{\pi})$

Let consider the terms $\mathbf{s}^s(\boldsymbol{\pi})$ and $\boldsymbol{\lambda}^s(\boldsymbol{\pi})$ defined in Proposition B1 with $\boldsymbol{\pi} \in \mathbb{R}^N$. Let define $\mathbf{u}_\ell$ as the *N* dimensional column vector with null elements with the exception of the ℓ^{th} one which is unitary. Let define $\mathbf{v}_\ell$ as the K dimensional column vector with null elements with the exception of the ℓ^{th} one which is unitary.

(*i*) Let $\eta_{mk} \in \mathbb{R}$. If $s^s_{mk}(\boldsymbol{\pi}) = 0$ then $s^s_{mk}(\boldsymbol{\pi} + \eta_{mk} \times \mathbf{u}_{K\times(k-1)+m}) = 0$ for $\eta_{mk} \le 0$. $s^s_{mk}(\boldsymbol{\pi})$ is strictly increasing in π_{mk} if $s^s_{mk}(\boldsymbol{\pi}) > 0$ or if $s^s_{mk}(\boldsymbol{\pi}) = 0 = \lambda^s_{mk}(\boldsymbol{\pi})$.

(*ii*) Let $\mu_m \in \mathbb{R}$. $d^s_m(\mu_m) \equiv \boldsymbol{\iota}'\mathbf{s}^s_{(m)}(\boldsymbol{\pi} - \mu_m \times \boldsymbol{\iota} \otimes \mathbf{v}_m)$ is strictly decreasing in μ_m if $d^s_m(\mu_m) > 0$. If $d^s_m(0) = 0$ then $d^s_m(\mu_m) = 0$ for $\mu_m \ge 0$.

(*iii*) Let $\mu_k \in \mathbb{R}$. $a^s_k(\mu_k) \equiv \boldsymbol{\iota}'\mathbf{s}^s_k(\boldsymbol{\pi} + \mu_k \times \mathbf{v}_k \otimes \boldsymbol{\iota})$ is strictly increasing in μ_k if $a^s_k(\mu_k) > 0$ or if $a^s_k(\mu_k) = 0 = \lambda^s_{mk}(\boldsymbol{\pi} + \mu_k \times \mathbf{u}_k \otimes \boldsymbol{\iota})$ for some $m \in \mathcal{K}$. If $\lambda^s_{mk}(\boldsymbol{\pi}) > 0$ for $m \in \mathcal{K}$ then $a^s_k(\mu_k) = 0$ for $\mu_m \le 0$.

Proof. The proof of this proposition mainly relies on results (*ii*) and (*v*) of Proposition B2. First note that $\mathbf{Z}_r \boldsymbol{\lambda}^s(\boldsymbol{\pi}) \ge \mathbf{0}$ is equivalent to:

$$\mathbf{Z}_r \boldsymbol{\pi} \le (\mathbf{Z}_r \mathbf{H} \mathbf{P}'_r \mathbf{G}_r)(\mathbf{P}_r \boldsymbol{\pi}) + \mathbf{L}_r \mathbf{h}.$$

This implies that if $r^s(\boldsymbol{\pi}) = r$ then $r^s(\boldsymbol{\pi} + \mathbf{Z}'_r \mathbf{p}) = r$ for any $\mathbf{p} \in \mathbb{R}_+^{N-\dim \mathcal{N}_r}$ and, as a result, this implies the second parts of results (*i*)-(*iii*) of Proposition C3.

Provided that $\mathbf{P}_r \mathbf{s}^s(\boldsymbol{\pi}) = -\mathbf{G}_r \mathbf{P}_r \mathbf{h} + \mathbf{G}_r(\mathbf{P}_r \boldsymbol{\pi})$ if $r^s(\boldsymbol{\pi}) = r$ and that $\mathbf{G}_r$ is positive definite we know that $s^s_{mk}(\boldsymbol{\pi})$ is strictly increasing in π_{mk} if $s^s_{mk}(\boldsymbol{\pi}) > 0$. If a frontier of $\mathcal{P}_r$ a point $\boldsymbol{\pi}$ such that $s^s_{mk}(\boldsymbol{\pi}) = 0 = \lambda^s_{mk}(\boldsymbol{\pi})$, then any increase in π_{mk} brings the considered point in $\operatorname{int} \mathcal{P}_r$ where $s^s_{mk}(\boldsymbol{\pi})$ is strictly increasing in π_{mk}. This implies that $s^s_{mk}(\boldsymbol{\pi})$ is also strictly increasing in π_{mk} if $s^s_{mk}(\boldsymbol{\pi}) = 0 = \lambda^s_{mk}(\boldsymbol{\pi})$.

By the definition of $d^s_m(\mu_m)$ we have $d^s_m(\mu_m) = (\boldsymbol{\iota} \otimes \mathbf{v}_m)\mathbf{s}^s(\boldsymbol{\pi} - \mu_m \times \boldsymbol{\iota} \otimes \mathbf{v}_m)$, and with $r^s(\boldsymbol{\pi}) = r$ and $\mathbf{s}^s(\boldsymbol{\pi}) = \mathbf{P}'_r \mathbf{G}_r \mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h})$, we obtain:

$$d_m^s(\mu_m) = (\boldsymbol{\iota} \otimes \mathbf{v}_m)\mathbf{P}_r'\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\pi} - \mathbf{h}) - \mu_m \times (\boldsymbol{\iota} \otimes \mathbf{v}_m)\mathbf{P}_r'\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\iota} \otimes \mathbf{v}_m)$$

or $d_m^s(\mu_m) = d_m^s(0) - \mu_m \times (\boldsymbol{\iota} \otimes \mathbf{v}_m)\mathbf{P}_r'\mathbf{G}_r\mathbf{P}_r(\boldsymbol{\iota} \otimes \mathbf{v}_m)$. Since $\mathbf{G}_r$ is positive definite, $d_m^s(\mu_m)$ is strictly increasing in μ_m if and only if $\mathbf{P}_r(\boldsymbol{\iota} \otimes \mathbf{v}_m) \neq \mathbf{0}$. It is easily shown that $\mathbf{P}_r(\boldsymbol{\iota} \otimes \mathbf{v}_m) \neq \mathbf{0}$ if and only if $s_{mk}^s(\boldsymbol{\pi}) > 0$ for some $k \in \mathcal{K}$, *i.e.* if and only if $d_m^s(\mu_m) = \boldsymbol{\iota}'\mathbf{s}_{(m)}^s(\boldsymbol{\pi} - \mu_m \times \boldsymbol{\iota} \otimes \mathbf{v}_m) > 0$.

The first part of result (*iii*) can be demonstrated by using a similar approach.

QED.

Appendix C. Myopic problem

This Appendix provides results related to the characterization of the myopic problem (M). Proposition 2 in the main text is defined as Proposition C1 in this Appendix.

Proposition C1. *Myopic problem*

Let consider the problem $\max_{\mathbf{s}\geq\mathbf{0}}\{\Pi(\mathbf{s};\boldsymbol{\pi})$ s.t. $\mathbf{Ds}=\mathbf{a}\}$ under the assumptions defined in Proposition B1. Let further assume that $\mathbf{a}\in\mathcal{A}\equiv\{\mathbf{a}\in\mathbb{R}_{+}^{K}:\boldsymbol{\iota}'\mathbf{a}>0\}$.

(*i*) The solution in $\mathbf{s}$ to problem (M) is unique and defines the function $\mathbf{s}^{m}:\mathbb{R}^{N}\times\mathcal{A}\rightarrow\mathbb{R}_{+}^{N}$ by

$\mathbf{s}^{m}(\boldsymbol{\pi},\mathbf{a})\equiv\arg\max_{\mathbf{s}\geq\mathbf{0}}\{\Pi(\mathbf{s};\boldsymbol{\pi})$ s.t. $\mathbf{Ds}=\mathbf{a}\}$. $\mathbf{s}^{m}$ is continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^{N}\times\mathcal{A}$.

(*ii*) The solution to problem (M) defines the value function $\Pi^{m}:\mathbb{R}^{N}\times\mathcal{A}\rightarrow\mathbb{R}$ by

$\Pi^{m}(\boldsymbol{\pi},\mathbf{a})\equiv\max_{\mathbf{s}\geq\mathbf{0}}\{\Pi(\mathbf{s};\boldsymbol{\pi})$ s.t. $\mathbf{Ds}=\mathbf{a}\}$. Π^{m} is convex in $\boldsymbol{\pi}$ and concave in $\mathbf{a}$ on $\mathbb{R}^{N}\times\mathcal{A}$.

(*iii*) $\mathbf{s}^{m}$ is piecewise affine and Π^{m} is piecewise quadratic in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^{N}\times\mathcal{A}$.

(*iv*) Π^{m} is continuously differentiable in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^{N}\times\mathcal{A}$ with $\frac{\partial}{\partial\boldsymbol{\pi}}\Pi^{m}(\boldsymbol{\pi},\mathbf{a})=\mathbf{s}^{m}(\boldsymbol{\pi},\mathbf{a})$ and

$\boldsymbol{\mu}^{m}(\boldsymbol{\pi},\mathbf{a})\equiv\frac{\partial}{\partial\mathbf{a}}\Pi^{m}(\boldsymbol{\pi},\mathbf{a})$.

Let define the Lagrangian problem associated to problem (M) as:

(LM) $\quad\min_{\boldsymbol{\mu},\boldsymbol{\lambda}\geq\mathbf{0}}\max_{\mathbf{s}}\{\Pi(\mathbf{s};\boldsymbol{\pi})+\mathbf{s}'\boldsymbol{\lambda}+\boldsymbol{\mu}'(\mathbf{a}-\mathbf{Ds})\}$

and let $\mathcal{M}^{m}(\boldsymbol{\pi},\mathbf{a})$ denote the set of solutions in $\boldsymbol{\mu}$ to problem (LM).

(*v*) $\mathbf{s}^{s}(\boldsymbol{\pi}-\mathbf{D}'\boldsymbol{\mu})=\mathbf{s}^{m}(\boldsymbol{\pi},\mathbf{a})$ and $\Pi^{s}(\boldsymbol{\pi}-\mathbf{D}'\boldsymbol{\mu})+\boldsymbol{\mu}'\mathbf{a}=\Pi^{m}(\boldsymbol{\pi},\mathbf{a})$ if and only if $\boldsymbol{\mu}\in\mathcal{M}^{m}(\boldsymbol{\pi},\mathbf{a})$.

(*vi*) $\mathcal{M}^{m}(\boldsymbol{\pi},\mathbf{a})=\arg\min_{\boldsymbol{\mu}\in\mathbb{R}^{K}}\{\Pi^{s}(\boldsymbol{\pi}-\mathbf{D}'\boldsymbol{\mu})+\boldsymbol{\mu}'\mathbf{a}\}$ and $\mathcal{M}^{m}(\boldsymbol{\pi},\mathbf{a})=\{\boldsymbol{\mu}\in\mathbb{R}^{K}:\mathbf{Ds}^{s}(\boldsymbol{\pi}-\mathbf{D}'\boldsymbol{\mu})=\mathbf{a}\}$.

(*vii*) Let define the sets $\mathcal{M}_n^m(\boldsymbol{\pi},\mathbf{a}) \equiv \{\mu_n \in \mathbb{R} : \boldsymbol{\mu} \in \mathcal{M}^m(\boldsymbol{\pi},\mathbf{a})\}$ for $n \in \mathcal{K}$. There exists a function $\boldsymbol{\mu}^m : \mathbb{R}^N \times \mathcal{A} \to \mathbb{R}^K$ such that:

(*a*) $\mathcal{M}_n^m(\boldsymbol{\pi},\mathbf{a}) \equiv \{\mu_n^m(\boldsymbol{\pi},\mathbf{a})\}$ if $a_n > 0$.

(*b*) $\mathcal{M}_n^m(\boldsymbol{\pi},\mathbf{a}) \equiv [\mu_n^m(\boldsymbol{\pi},\mathbf{a}),+\infty)$ if $a_n = 0$.

(*c*) $\frac{\partial}{\partial \mathbf{a}} \Pi^m(\boldsymbol{\pi},\mathbf{a}) = \boldsymbol{\mu}^m(\boldsymbol{\pi},\mathbf{a})$ for any $(\boldsymbol{\pi},\mathbf{a}) \in \mathbb{R}^N \times \mathcal{A}$.

(*d*) $\boldsymbol{\mu}^m$ is continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N \times \mathcal{A}$.

Proof. The results collected in (*i*)-(*iii*) are either well-known or demonstrated in Bemporad *et al* (2002). Result (*iv*), stating that Π^m is continuously differentiable in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N \times \mathcal{A}$, is to be demonstrated together with result (*vii*). The result stating that Π^m is continuously differentiable in $\mathbf{a}$ on $\mathbb{R}^N \times \mathcal{A}$ relies on a specific approach. Note that the results collected in Proposition C2, which is given below, are closely linked to result (*vii*).

The proof of results (*v*) and (*vi*) rely on the functional form of the Lagrangian problem (LM) associated to problem (M) and on the links of problems (LM) and (S). Let first remark that the Lagrangian function associated to problem (M) satisfies:

$$\Pi(\mathbf{s};\boldsymbol{\pi}) + \boldsymbol{\lambda}'\mathbf{s} + \boldsymbol{\mu}'(\mathbf{a} - \mathbf{D}\mathbf{s}) = \Pi(\mathbf{s};\boldsymbol{\pi} - \mathbf{D}'\boldsymbol{\mu}) + \boldsymbol{\lambda}'\mathbf{s} + \boldsymbol{\mu}'\mathbf{a}.$$

The term $\boldsymbol{\lambda}$ is the Lagrange multiplier vector associated to the non-negativity constraint $\mathbf{s} \geq \mathbf{0}$ and $\boldsymbol{\mu}$ is the Lagrange multiplier vector associated to the crop rotation constraint $\mathbf{D}\mathbf{s} = \mathbf{a}$. Result (*v*) then makes uses of the fact that problem (LM) can be decomposed as:

$$\min_{\boldsymbol{\mu}}\{\min_{\boldsymbol{\lambda} \geq \mathbf{0}}\{\max_{\mathbf{s}} \Pi(\mathbf{s};\boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu}) + \boldsymbol{\lambda}'\mathbf{s}\} + \boldsymbol{\mu}'\mathbf{a}\}.$$

The KKT conditions associated to this Lagrangian problem, *i.e.*

$$\begin{cases} \boldsymbol{\pi} - \mathbf{h} + \boldsymbol{\lambda} - \boldsymbol{\iota} \otimes \boldsymbol{\mu} - \mathbf{Hs} = \mathbf{0} \\ \mathbf{Ds} - \mathbf{a} = \mathbf{0} \\ \boldsymbol{\lambda}'\mathbf{s} = 0 \text{ with } \mathbf{s} \geq \mathbf{0} \text{ and } \boldsymbol{\lambda} \geq \mathbf{0} \end{cases},$$

characterize the solutions in $(\mathbf{s}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ to this problem. Problem (LM) is also equivalent to the following modified Lagrangian problem

$$\min_{\boldsymbol{\mu}} \{\max_{\mathbf{s} \geq \mathbf{0}} \Pi(\mathbf{s}; \boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu}) + \boldsymbol{\mu}'\mathbf{a}\}$$

and, by using Proposition B1, to the following dual problem:

$$\min_{\boldsymbol{\mu}} \{\Pi^{s}(\boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu}) + \boldsymbol{\mu}'\mathbf{a}\} .$$

This allows rewriting the KKT conditions of Problem (LM) as:

$$\begin{cases} \mathbf{s}^{m}(\boldsymbol{\pi}, \mathbf{a}) = \mathbf{s}^{s}(\boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu}) \\ \mathbf{D}\mathbf{s}^{s}(\boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu}) = \mathbf{a} \end{cases}$$

provided that $\boldsymbol{\lambda}$ is the optimal LM vector of the non-negativity constraints of Lagrangian problem associated to the static problem $\max_{\mathbf{s} \geq \mathbf{0}} \Pi(\mathbf{s}; \boldsymbol{\pi} - \boldsymbol{\iota} \otimes \boldsymbol{\mu})$. These equivalences provide results (*v*) and (*vi*).

Before proving results (*iv*) and (*vii*) let consider the case where $a_n > 0$ for $n \in \mathcal{K}$. With $a_n > 0$ for $n \in \mathcal{K}$ implies that the (linear) constraints involved in problem (M) which are active at the optimum cannot be redundant, *i.e*. are linearly independent. The condition $a_n > 0$ ensures that $s_{nk}^{m}(\boldsymbol{\pi}, \mathbf{a}) > 0$ for some $k \in \mathcal{K}$ and, as a result, implies that the non-negativity constraint $s_{nk} \geq 0$ is (strongly) inactive. Hence the condition stating that $a_n > 0$ for $n \in \mathcal{K}$ that the LICQ condition holds for any solution to problem (LM). The objective function of this maximization problem being strictly concave and twice continuously differentiable, and its constraints being linear, the results due to Jittorntrum (1984) reported as Theorems 4.1 and

7.3 in Fiacco and Kyparisis (1985) is applicable. These results give that Π^m is continuously differentiable in $(\boldsymbol{\pi}, \mathbf{a})$ on $\mathbb{R}^N \times \mathcal{A}^+$ where $\mathcal{A}^+ \equiv \{\mathbf{a} \in \mathbb{R}^K_{++} \mid \mathbf{a}'\boldsymbol{\iota} > 0\}$.

If $a_n = 0$ for some $n \in \mathcal{K}$ or if $\mathbf{a} \in \mathcal{A} \setminus \mathcal{A}^+$, the LICQ condition doesn't hold for the solutions to problem (LM), implying multiple solutions in elements of the Lagrange multiplier vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. If $a_n = 0$ then the crop rotation constraint $\boldsymbol{\iota}'\mathbf{s}_{(m)} = a_n$ and the non-negativity constraint $\mathbf{s}_{(m)} \geq \mathbf{0}$ imply $\mathbf{s}_{(m)} = \mathbf{0}$, *i.e.* the constraint $\mathbf{s}_{(m)} \geq \mathbf{0}$ is active and redundant with $\boldsymbol{\iota}'\mathbf{s}_{(m)} = a_n$ at the optimum of problems (M) and (LM).

We do not proceed by eliminating redundant constraints because this approach is not well suited for dealing with dynamic programming problems. We use results due to Berkelaar *et al* (1997). These authors investigated the differentiability properties of the value function of general quadratic programming problems.

When applied to Π^m for derivatives in $\boldsymbol{\pi}$ Theorem 58 of Berkelaar *et al* (1997) provides:

$$\tfrac{\partial}{\partial \pi_n^+} \Pi^m(\boldsymbol{\pi}, \mathbf{a}) = \min_{\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}} \{s_n \text{ s.t. } (\mathbf{s}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{L}(\boldsymbol{\pi}, \mathbf{a})\}$$

and:

$$\tfrac{\partial}{\partial \pi_n^-} \Pi^m(\boldsymbol{\pi}, \mathbf{a}) = \max_{\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}} \{s_n \text{ s.t. } (\mathbf{s}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathcal{L}(\boldsymbol{\pi}, \mathbf{a})\}$$

where $\mathcal{L}(\boldsymbol{\pi}, \mathbf{a})$ is the subset of $\mathbb{R}^{2N}_+ \times \mathbb{R}^K$ containing the solutions in $(\mathbf{s}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ to the KKT conditions. Provided that $\mathbf{s}^m(\boldsymbol{\pi}, \mathbf{a})$ is the unique solution in $\mathbf{s}$ to problem (M) we have:

$$\tfrac{\partial}{\partial \pi_n^-} \Pi^m(\boldsymbol{\pi}, \mathbf{a}) = \tfrac{\partial}{\partial \pi_n^+} \Pi^m(\boldsymbol{\pi}, \mathbf{a}) = s_n^m(\boldsymbol{\pi}, \mathbf{a}) = \tfrac{\partial}{\partial \pi_n} \Pi^m(\boldsymbol{\pi}, \mathbf{a}),$$

i.e. Π^m is differentiable in $\boldsymbol{\pi}$ on $\mathbb{R}^N \times \mathcal{A}$. Given that $\mathbf{s}^m(\boldsymbol{\pi}, \mathbf{a})$ is continuous in $\boldsymbol{\pi}$ on $\mathbb{R}^N \times \mathcal{A}$, Π^m is continuously differentiable in $\boldsymbol{\pi}$ on $\mathbb{R}^N \times \mathcal{A}$.

When applied to Π^m for derivatives in a_n Theorem 50 of Berkelaar *et al* (1997) yields:

$$\frac{\partial}{\partial a_n^+}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \min_{\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}}\{\mu_n \text{ s.t. } (\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{L}(\boldsymbol{\pi},\mathbf{a})\}$$

and:

$$\frac{\partial}{\partial a_n^-}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \max_{\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}}\{\mu_n \text{ s.t. } (\mathbf{s},\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{L}(\boldsymbol{\pi},\mathbf{a})\}.$$

These results can be further specialized by using the uniqueness of the solution in **s** to problem (M). We have $\mathcal{L}(\boldsymbol{\pi},\mathbf{a}) = \{\mathbf{s}^m(\boldsymbol{\pi},\mathbf{a})\}\times\mathcal{Q}(\boldsymbol{\pi},\mathbf{a})$ and the first equation of the KKT condition system, *i.e.* $\boldsymbol{\pi}+\boldsymbol{\lambda}-\boldsymbol{\iota}\otimes\boldsymbol{\mu} = \mathbf{h}+\mathbf{H}\mathbf{s}^m(\boldsymbol{\pi},\mathbf{a})$, ensures that the difference $\boldsymbol{\lambda}-\boldsymbol{\iota}\otimes\boldsymbol{\mu}$ is also unique for any $(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{Q}(\boldsymbol{\pi},\mathbf{a})$. With $\boldsymbol{\eta}^m(\boldsymbol{\pi},\mathbf{a}) \equiv \boldsymbol{\pi}-\mathbf{h}-\mathbf{H}\mathbf{s}^m(\boldsymbol{\pi},\mathbf{a})$ we have $\boldsymbol{\iota}\otimes\boldsymbol{\mu}-\boldsymbol{\lambda} = \boldsymbol{\eta}^m(\boldsymbol{\pi},\mathbf{a})$ if and only if $(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{Q}(\boldsymbol{\pi},\mathbf{a})$. Note also that $\boldsymbol{\eta}^m : \mathbb{R}^N\times\mathcal{A}\to\mathbb{R}^N$ is continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N\times\mathcal{A}$ by the continuity of $\mathbf{s}^m$. Let use the upper index "*m*" for indicating an element of $\mathcal{Q}(\boldsymbol{\pi},\mathbf{a})$, *i.e.* $(\boldsymbol{\lambda}^m,\boldsymbol{\mu}^m)\in\mathcal{Q}(\boldsymbol{\pi},\mathbf{a})$ and $\boldsymbol{\iota}\otimes\boldsymbol{\mu}^m-\boldsymbol{\lambda}^m = \boldsymbol{\eta}^m(\boldsymbol{\pi},\mathbf{a})$. "Fixing" the value of **s** at $\mathbf{s}^m(\boldsymbol{\pi},\mathbf{a})$ in the KKT condition system we obtain:

$$\frac{\partial}{\partial a_n^+}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \min_{(\mu_n,\boldsymbol{\lambda}_{(n)})}\{\mu_n \text{ s.t. } (\mu_n,\boldsymbol{\lambda}_{(n)}) \in \mathcal{Q}_n(\boldsymbol{\pi},\mathbf{a})\}$$

and:

$$\frac{\partial}{\partial a_n^-}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \max_{(\mu_n,\boldsymbol{\lambda}_{(n)})}\{\mu_n \text{ s.t. } (\mu_n,\boldsymbol{\lambda}_{(n)}) \in \mathcal{Q}_n(\boldsymbol{\pi},\mathbf{a})\}$$

where:

$$\mathcal{Q}_n(\boldsymbol{\pi},\mathbf{a}) \equiv \left\{(\mu_n,\boldsymbol{\lambda}_{(n)})\in\mathbb{R}\times\mathbb{R}_+^K \text{ s.t. } \left| \begin{array}{c} \mu_n-\lambda_{nk} = \eta_{nk}^m(\boldsymbol{\pi},\mathbf{a}),\ \lambda_{nk}s_{nk}^m(\boldsymbol{\pi},\mathbf{a}) = 0 \text{ and } \lambda_{nk}\geq 0 \\ \text{for } k\in\mathcal{K} \end{array}\right.\right\}.$$

For later use let define the term $\overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a}) \equiv \max\{\eta_{nk}^m(\boldsymbol{\pi},\mathbf{a}) : k\in\mathcal{K}\}$. Given that $\boldsymbol{\eta}^m$ is continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N\times\mathcal{A}$, $\overline{\eta}_n^m \equiv \max\{\eta_{nk}^m : k\in\mathcal{K}\}$ is also continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N\times\mathcal{A}$.

Two cases occur for $\boldsymbol{\lambda}_{(n)}^m$ and μ_n^m, depending on whether $a_n > 0$ or $a_n = 0$.

If $a_n > 0$ then there exists $\ell \in \mathcal{K}$ such that $s_{n\ell}^m(\boldsymbol{\pi},\mathbf{a}) > 0$. This implies that $\lambda_{n\ell}^m = 0$ and, as a result, that $\mu_n^m = \eta_{n\ell}^m(\boldsymbol{\pi},\mathbf{a})$. Hence the solution in μ_n to problem (LM) is unique if $a_n > 0$. Let denote this solution by $\mu_n^m(\boldsymbol{\pi},\mathbf{a})$. This implies that:

$$\tfrac{\partial}{\partial a_n^-}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \tfrac{\partial}{\partial a_n^+}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \tfrac{\partial}{\partial a_n}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \mu_n^m(\boldsymbol{\pi},\mathbf{a}) \text{ if } a_n > 0.$$

Provided that $\lambda_{n\ell}^m = 0$ if and only if $\mu_n^m(\boldsymbol{\pi},\mathbf{a}) = \eta_{n\ell}^m(\boldsymbol{\pi},\mathbf{a})$, we also have $\mu_n^m(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})$, *i.e.*:

$$\tfrac{\partial}{\partial a_n}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \mu_n^m(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a}) \text{ if } a_n > 0.$$

If $a_n = 0$ then $\mathbf{s}_n^m(\boldsymbol{\pi},\mathbf{a}) = \mathbf{0}$ and, $\boldsymbol{\lambda}_{(n)}^m$ and μ_n^m are not uniquely characterized by the KKT conditions. But $\boldsymbol{\lambda}_{(n)}^m \geq \mathbf{0}$ implies that $\mu_n^m \geq \eta_{nk}^m(\boldsymbol{\pi},\mathbf{a})$ for any $k \in \mathcal{K}$, *i.e.* that μ_n^m is bounded from below with:

$$\min_{(\mu_n,\boldsymbol{\lambda}_{(n)})}\{\mu_n \text{ s.t. } (\mu_n,\boldsymbol{\lambda}_{(n)}) \in \mathcal{Q}_n(\boldsymbol{\pi},\mathbf{a})\} = \min\{\mu_n^m\} = \max\{\eta_{nk}^m(\boldsymbol{\pi},\mathbf{a}) : k \in \mathcal{K}\} = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a}).$$

As a result we have:

$$\tfrac{\partial}{\partial a_n^+}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \mu_n^m(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a}) \text{ if } a_n = 0.$$

Provided that $\mu_n^m(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})$ is continuous in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N \times \mathcal{A}$ we have:

$$\tfrac{\partial}{\partial a_n}\Pi^m(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a}) = \mu_n^m(\boldsymbol{\pi},\mathbf{a}) \text{ for any } a_n \geq 0$$

and it follows that $\Pi^m(\boldsymbol{\pi},\mathbf{a})$ is continuously differentiable in $(\boldsymbol{\pi},\mathbf{a})$ on $\mathbb{R}^N \times \mathcal{A}$.

QED.

The intuition of this result is as follows. The term $\overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})$ allows the identification of the (n,k) pairs for which $a_n=0$ is the least constraining. Basically, to make available an infinitesimal acreage preceding crop *n*, $\varepsilon_n>0$, implies that the optimal choice of s_{nk} remains null for any crop $k\in\mathcal{K}$ such that $\eta_{nk}^m(\boldsymbol{\pi},\mathbf{a})<\overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})$ while that of $s_{n\ell}$ becomes strictly positive for any crop $\ell\in\mathcal{K}$ such that $\eta_{n\ell}^m(\boldsymbol{\pi},\mathbf{a})=\overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})$. With

$$\mu_n^m(\boldsymbol{\pi},\mathbf{a})=\overline{\eta}_n^m(\boldsymbol{\pi},\mathbf{a})=\tfrac{\partial}{\partial a_n}\Pi^m(\boldsymbol{\pi},\mathbf{a}),$$

$\mu_n^m(\boldsymbol{\pi},\mathbf{a})$ is the maximum renting price that the considered farmer would pay for increasing his acreage of land with preceding crop *n* when such land is unavailable on his farm.

Proposition C2. *Constrained myopic problem*

Let assume that $\mathcal{K}^+(\mathbf{a})\subset\mathcal{K}$ and let consider the “constrained myopic” problem:

(C) $\quad \max_{\mathbf{s}\geq\mathbf{0}}\{\Pi(\mathbf{s};\boldsymbol{\pi})$ s.t. $\boldsymbol{\iota}'\mathbf{s}_{(n)}=a_n$ for $n\in\mathcal{K}^+(\mathbf{a})$ and $\mathbf{s}_{(n)}=\mathbf{0}$ for $n\in\mathcal{K}^0(\mathbf{a})\}$

under the assumptions collected in Proposition C1. Let consider the corresponding Lagrangian problem:

$$\text{(LC)}\quad \min_{\substack{\boldsymbol{\eta}_{(n)},n\in\mathcal{K}^0(\mathbf{a})\\ \mu_n,\boldsymbol{\lambda}_{(n)}\geq\mathbf{0},n\in\mathcal{K}^+(\mathbf{a})}}\ \max_{\mathbf{s}}\left\{\Pi(\mathbf{s};\boldsymbol{\pi})+\sum\nolimits_{n\in\mathcal{K}^+(\mathbf{a})}\left(\boldsymbol{\lambda}'_{(n)}\mathbf{s}_{(n)}+\mu_n(a_n-\boldsymbol{\iota}'\mathbf{s}_{(n)})\right)-\sum\nolimits_{n\in\mathcal{K}^0(\mathbf{a})}\boldsymbol{\eta}'_{(n)}\mathbf{s}_{(n)}\right\}.$$

(*i*) The solutions in $\mathbf{s}$, $\boldsymbol{\eta}_{(n)}$ for $n\in\mathcal{K}^0(\mathbf{a})$, μ_n and $\boldsymbol{\lambda}_{(n)}$ for $n\in\mathcal{K}^+(\mathbf{a})$ to problem (LC) are unique. They define the functions $\mathbf{s}^c(\boldsymbol{\pi},\mathbf{a})$, $\boldsymbol{\eta}_{(n)}^c(\boldsymbol{\pi},\mathbf{a})$ for $n\in\mathcal{K}^0(\mathbf{a})$, $\mu_n^c(\boldsymbol{\pi},\mathbf{a})$ and $\boldsymbol{\lambda}_{(n)}^c(\boldsymbol{\pi},\mathbf{a})$ for $n\in\mathcal{K}^+(\mathbf{a})$.

Let denote the value function of problem (C) by $\Pi^c(\boldsymbol{\pi},\mathbf{a})$. We have:

(*ii*) $\tfrac{\partial}{\partial\boldsymbol{\pi}}\Pi^c(\boldsymbol{\pi},\mathbf{a})=\mathbf{s}^c(\boldsymbol{\pi},\mathbf{a})=\mathbf{s}^m(\boldsymbol{\pi},\mathbf{a})$.

(*iii*) $\tfrac{\partial}{\partial a_n}\Pi^c(\boldsymbol{\pi},\mathbf{a})=\mu_n^c(\boldsymbol{\pi},\mathbf{a})=\mu_n^m(\boldsymbol{\pi},\mathbf{a})$ and $\boldsymbol{\lambda}_{(n)}^c(\boldsymbol{\pi},\mathbf{a})=\boldsymbol{\lambda}_{(n)}^m(\boldsymbol{\pi},\mathbf{a})$ if $n\in\mathcal{K}^+(\mathbf{a})$.

(*iv*) $\boldsymbol{\eta}^{c}_{(n)}(\boldsymbol{\pi},\mathbf{a}) = \mu^{m}_{n} \times \boldsymbol{\iota} - \boldsymbol{\lambda}^{m}_{(n)}$ where $(\boldsymbol{\lambda}^{m}_{(n)}, \mu^{m}_{n})$ is any solution in $(\boldsymbol{\lambda}_{(n)}, \mu_{n})$ to problem (LM) if $n \in \mathcal{K}^{0}(\mathbf{a})$.

(*v*) $\mu^{m}_{n}(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}^{c}_{n}(\boldsymbol{\pi},\mathbf{a}) \equiv \max\{\eta^{c}_{nk}(\boldsymbol{\pi},\mathbf{a}) : k \in \mathcal{K}\}$ for $n \in \mathcal{K}^{0}(\mathbf{a})$.

Proof. Problem (C) is a quadratic programming problem with non redundant linear constraints. Its solution in $\mathbf{s}$ are unique and its KKT conditions uniquely characterize the optimal Lagrange multiplier vectors $\boldsymbol{\eta}^{c}_{(n)}(\boldsymbol{\pi},\mathbf{a})$ for $n \in \mathcal{K}^{0}(\mathbf{a})$ on the one hand and, $\mu^{c}_{n}(\boldsymbol{\pi},\mathbf{a})$ and $\boldsymbol{\lambda}^{c}_{(n)}(\boldsymbol{\pi},\mathbf{a})$ for $n \in \mathcal{K}^{+}(\mathbf{a})$ on the other hand. This provides result (*i*). The KKT conditions of problem (C) are identical to those of problem (M) as soon as we consider $\boldsymbol{\eta}^{m}_{(n)}(\boldsymbol{\pi},\mathbf{a}) \equiv \mu^{m}_{n} \times \boldsymbol{\iota} - \boldsymbol{\lambda}^{m}_{(n)}$ instead of both μ^{m}_{n} and $\boldsymbol{\lambda}^{m}_{(n)}$ for $n \in \mathcal{K}^{0}(\mathbf{a})$. This provides results (*ii*)-(*iv*). Result (*v*) is obtained by using the definition of $\mu^{m}_{n}(\boldsymbol{\pi},\mathbf{a})$ in Proposition C1.

QED.

These results related to cases where some previous crop(s) is(are) unavailable on the farm can be summarized by considering problems equivalent to problem (M). Let assume that $a_{n} = 0$ and let consider the maximization problem equivalent to problem (M) obtained by replacing the constraints $\mathbf{s}_{(n)} \geq \mathbf{0}$ and $\boldsymbol{\iota}'\mathbf{s}_{(n)} = a_{n}$ by the constraint $\mathbf{s}_{(n)} = \mathbf{0}$. The optimal Lagrange multiplier vector associated to this equality constraint, $\boldsymbol{\eta}_{(n)}(\boldsymbol{\pi},\mathbf{a}) \equiv (\eta_{nk}(\boldsymbol{\pi},\mathbf{a}) : k \in \mathcal{K})$, is unique and allows characterizing $\mu^{m}_{n}(\boldsymbol{\pi},\mathbf{a})$ with $\mu^{m}_{n}(\boldsymbol{\pi},\mathbf{a}) = \overline{\eta}_{n}(\boldsymbol{\pi},\mathbf{a})$ where $\overline{\eta}_{n}(\boldsymbol{\pi},\mathbf{a}) \equiv \max\{\eta_{nk}(\boldsymbol{\pi},\mathbf{a}) : k \in \mathcal{K}\}$. This result has two main implications. First, it implies that $\mu^{m}_{n}(\boldsymbol{\pi},\mathbf{a})$ is easily computed by enforcing the constraint $\mathbf{s}_{(n)} = \mathbf{0}$ for $n \in \mathcal{K}$ such that $a_{n} = 0$ in problem (M). However, if this approach is convenient for solving myopic problems, it is much less relevant for characterizing the solutions to dynamic acreage choice problems. Second, this result is insightful for analyzing the mechanisms lying at the root of the characterization of $\mu_n^m(\boldsymbol{\pi},\mathbf{a})$. The equality $\mu_n^m(\boldsymbol{\pi},\mathbf{a})=\overline{\eta}_n(\boldsymbol{\pi},\mathbf{a})$ states that he marginal effect of an increase in a_n from 0 on $\Pi^m(\boldsymbol{\pi},\mathbf{a})$, *i.e.* $\mu_n^m(\boldsymbol{\pi},\mathbf{a})$, is equal to the optimal Lagrange multiplier associated to the least constrained crops when the constraints $s_{nk}=0$ are imposed for $k\in\mathcal{K}$, *i.e.* $\overline{\eta}_n(\boldsymbol{\pi},\mathbf{a})$. Let assume that the farmer obtains a very small acreage with preceding crop *n*. His optimal choice consists in devoting this acreage to the crop sequence(s) (n,ℓ) for $\ell\in\mathcal{K}$ such that $\eta_{n\ell}(\boldsymbol{\pi},\mathbf{a})=\overline{\eta}_n(\boldsymbol{\pi},\mathbf{a})$. $\overline{\eta}_n(\boldsymbol{\pi},\mathbf{a})$ is positive if it is profitable for the farmer to grow crop(s) ℓ after crop *n*. $\overline{\eta}_n(\boldsymbol{\pi},\mathbf{a})$ is negative if the farmer would be forced to grow crop(s) ℓ after crop *n*, *i.e.* if the farmer would not rent any acreage with preceding crop *n*.

Appendix D. Dynamic problem

This Appendix provides results related to the characterization of the dynamic problem (D). Proposition 2 in the main text is defined as Proposition D1 in this Appendix. It characterizes the solutions to the dynamic programming problem considered in the article by adopting a stochastic programming approach, *i.e.* by directly solving the dynamic problem as a large static problem while taking advantage of its multistage structure.

Proposition D2 provides additional results by adopting a dynamic programming approach, *i.e.* by relying on Bellman's dynamic programming principle. Proposition D3 provides detailed results on the differentiability properties of the value functions of the considered dynamic problem.

Proposition D1. *Dynamic problem, stochastic programming approach*

Let define the vector of contingent acreage choices as $\mathbf{s} \equiv (\mathbf{s}_{(t)} : t = 1,...,T)$ with $\mathbf{s}_{(t)} \equiv (\mathbf{s}_{\omega(t)} : \omega_t \in \mathcal{W}_t)$ for $t = 1,...,T$ and let define the support point vector of the crop gross margin vector as $\bar{\boldsymbol{\pi}} \equiv (\bar{\boldsymbol{\pi}}_{(t)} : t = 1,...,T)$ with $\bar{\boldsymbol{\pi}}_{(t)} \equiv (\bar{\boldsymbol{\pi}}_{t|\omega_t} : \omega_t \in \mathcal{W}_t)$ for $t = 1,...,T$.

Let consider problem (D^d): $\max_{\mathbf{s} \in \mathcal{F}(\mathbf{a}_0)} \sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t})$ where the feasible set $\mathcal{F}(\mathbf{a}_0)$ is defined as:

$$\mathcal{F}(\mathbf{a}_0) \equiv \{\mathbf{s} \in \mathbb{R}_+^{NW} : \mathbf{D}\mathbf{s}_{\omega_1} = \mathbf{a}_0 \text{ and } \mathbf{D}\mathbf{s}_{\omega_t} = \mathbf{A}\mathbf{s}_{\omega_{t-1}} \text{ for } \omega_t \in \mathcal{W}_t(\omega_{t-1}),\ \omega_{t-1} \in \mathcal{W}_{t-1},\ t = 2,...,T\}$$

under the assumptions stated in Propositions B1 and C1.

(*i*) The solution in $\mathbf{s}$ to problem (D^d) is unique. Let $\mathbf{s}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ denote this solution with

$$\mathbf{s}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv (\mathbf{s}_{(t)}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) : t = 1,...,T) \text{ and } \mathbf{s}_{(t)}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv (\mathbf{s}_{\omega_t}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) : \omega_t \in \mathcal{W}_t).$$

(ii) The function $\mathbf{s}^o : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}_+^{NW}$ defined by the solution in $\mathbf{s}$ to problem (D^d)

$\mathbf{s}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \arg\max_{\mathbf{s}\in\mathcal{F}(\mathbf{a}_0)} \sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t})$ is piecewise affine and continuous in $(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$.

(iii) The value function $V_{\omega_1}^o : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}^{NW}$ of problem (D^d) defined by

$V_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \max_{\mathbf{s}\in\mathcal{F}(\mathbf{a}_0)} \sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t})$ is piecewise quadratic and continuous in $(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$. $V_{\omega_1}^o$ is convex in $\bar{\boldsymbol{\pi}}$ and concave in $\mathbf{a}_0$ on $\mathbb{R}^{NW} \times \mathcal{A}$.

Let $\boldsymbol{\mu} \in \mathbb{R}^{KW}$ denote a vector with the structure of $\mathbf{s}$, *i.e.* $\boldsymbol{\mu} \equiv (\boldsymbol{\mu}_{(t)} : t = 1,...,T)$ with $\boldsymbol{\mu}_{(t)} \equiv (\boldsymbol{\mu}_{\omega_t} : \omega_t \in \mathcal{W}_t)$. Let define the term $\bar{\boldsymbol{\mu}}_{t+1|\omega_t}$ by $\bar{\boldsymbol{\mu}}_{t+1|\omega_t} \equiv \sum_{\omega_{t+1} \in \mathcal{W}_{t+1}(\omega_t)} p_{\omega_{t+1}|\omega_t} \boldsymbol{\mu}_{\omega_{t+1}}$ with the convention $\boldsymbol{\mu}_{\omega_{T+1}} \equiv \mathbf{0}$.

(iv) To solve problem (D^d) is equivalent to solve the following dual problem:

(DD^d) $$\min_{\boldsymbol{\mu}} \left\{ \sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi^s(\bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t}) + \boldsymbol{\mu}'_{\omega_1}\mathbf{a}_0 \right\}$$

with:

$$\mathbf{s}_{\omega_t}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) = \mathbf{s}^s(\bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t}) \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,...,T$$

for any $\boldsymbol{\mu} \in \mathcal{M}_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ where $\mathcal{M}_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ is the solution set in to problem (DD^d), *i.e.*:

$$\mathcal{M}_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \arg\min_{\boldsymbol{\mu}} \left\{ \sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi^s(\bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t}) + \boldsymbol{\mu}'_{\omega_1}\mathbf{a}_0 \right\}.$$

(v) $\boldsymbol{\mu} \in \mathcal{M}_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$ if and only if $\boldsymbol{\mu}$ is a solution to equation system:

$$\begin{cases} \mathbf{D}\mathbf{s}^s(\bar{\boldsymbol{\pi}}_{1|\omega_1} - \mathbf{D}'\boldsymbol{\mu}_{\omega_1} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{2|\omega_1}) = \mathbf{a}_0 \\ \mathbf{D}\mathbf{s}^s(\bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t}) = \mathbf{A}\mathbf{s}^s(\bar{\boldsymbol{\pi}}_{t-1|\omega_{t-1}} - \mathbf{D}'\boldsymbol{\mu}_{\omega_{t-1}} + \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t|\omega_{t-1}}) \\ \text{for } \omega_{t-1} \in \mathcal{W}_{t-1},\ \omega_t \in \mathcal{W}_t(\omega_{t-1}) \text{ and } t = 2,...,T. \end{cases}$$

(vi) $V_{\omega_1}^o$ is continuously differentiable in $\bar{\boldsymbol{\pi}}$ on $\mathbb{R}^{NW} \times \mathcal{A}$ with $\frac{\partial}{\partial \bar{\boldsymbol{\pi}}} V_{\omega_1}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) = \mathbf{s}^o(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$.

(*vii*) $V^o_{\omega_1}$ is continuously differentiable in $\mathbf{a}_0$ on $\mathbb{R}^{NW} \times \mathcal{A}$. *I.e.*, there exists a function

$\boldsymbol{\mu}^o_{\omega_1} : \mathbb{R}^{NW} \times \mathcal{A} \rightarrow \mathcal{M}^o_{\omega_1}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$, continuous in $\mathbf{a}_0$ on $\mathbb{R}^{NW} \times \mathcal{A}$, with $\frac{\partial}{\partial \mathbf{a}_0} V^o_{\omega_1}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) = \boldsymbol{\mu}^o_{\omega_1}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)$.

Proof. Provided that Π is quadratic and strictly convex in each $\mathbf{s}_{\omega_t}$, that $\beta > 0$ and that $p_{\omega_t} > 0$, the objective function of problem (D^d), $\sum_{t=1}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_t} p_{\omega_t} \Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t})$, is strictly convex in **s**. It is easily shown that the feasible set $\mathcal{F}(\mathbf{a}_0)$ is a polyhedron of $\mathbb{R}^{NW}_+$ with a non-empty interior. These conditions imply that problem (D^d) is a (possibly very large) strictly convex quadratic programming problem with strongly feasible linear constraints. This in turn implies that results (*i*)-(*iii*) can be obtained by applying the results of Bemporad *et al* (2002).

Results (*iv*)-(*vi*) are obtained by using a Lagrangian approach with a specific Lagrangian function. The device used here consists in discounting, by β^{t-1}, as well as in weighting, by p_{ω_t}, the Lagrange multiplier vectors $\boldsymbol{\lambda}_{\omega_t}$ and $\boldsymbol{\mu}_{\omega_t}$. Indeed, this device simply consists in defining the Lagrangian problem associated to problem (D^d) according to its multistage structure, as in the dynamic programming approach. The considered Lagrangian problem is defined as:

$$\min_{\boldsymbol{\mu}} \min_{\boldsymbol{\lambda} \geq \mathbf{0}} \max_{\mathbf{s}} L(\mathbf{s}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \bar{\boldsymbol{\pi}}, \mathbf{a}_0)$$

where:

$$\begin{aligned} L(\mathbf{s}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv {} & \Pi(\mathbf{s}_{\omega_1}; \bar{\boldsymbol{\pi}}_{1|\omega_1}) + \mathbf{s}'_{\omega_1} \boldsymbol{\lambda}_{\omega_1} + \boldsymbol{\mu}'_{\omega_1}(\mathbf{a}_0 - \mathbf{D}\mathbf{s}_{\omega_1}) \\ & + \sum_{t=2}^{T} \beta^{t-1} \sum_{\omega_{t-1} \in \mathcal{W}_{t-1}} \sum_{\omega_t \in \mathcal{W}_{t|\omega_{t-1}}} p_{\omega_t} \{\Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t}) + \mathbf{s}'_{\omega_t} \boldsymbol{\lambda}_{\omega_t}\} \\ & + \sum_{t=2}^{T} \beta^{t-1} \sum_{\omega_{t-1} \in \mathcal{W}_{t-1}} p_{\omega_{t-1}} \sum_{\omega_t \in \mathcal{W}_{t|\omega_{t-1}}} p_{\omega_t|\omega_{t-1}} \boldsymbol{\mu}'_{\omega_t}(\mathbf{A}\mathbf{s}_{\omega_{t-1}} - \mathbf{D}\mathbf{s}_{\omega_t}) \end{aligned}$$

and where the non-negativity constraint Lagrange multiplier vector $\boldsymbol{\lambda} \in \mathbb{R}^{NW}_+$ has the structure of **s**, *i.e.* $\boldsymbol{\lambda} \equiv (\boldsymbol{\lambda}_{(t)} : t = 1, ..., T)$ with $\boldsymbol{\lambda}_{(t)} \equiv (\boldsymbol{\lambda}_{\omega_t} : \omega_t \in \mathcal{W}_t)$.

The feasible set having a non-empty interior, problem (D^d) is strongly dual and the solutions to the Lagrangian problem provide those to the original one, *e.g.* we have:

$$V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\min_{\boldsymbol{\mu}}\min_{\boldsymbol{\lambda}\geq\mathbf{0}}\max_{\mathbf{s}}L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0).$$

Rearranging the terms in $L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ yields:

$$L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\sum_{t=1}^{T}\beta^{t-1}\sum_{\omega_t\in\mathcal{W}_t}p_{\omega_t}\{\Pi(\mathbf{s}_{\omega_t};\bar{\boldsymbol{\pi}}_{t|\omega_t}-\mathbf{D}'\boldsymbol{\mu}_{\omega_t}+\beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t})+\mathbf{s}'_{\omega_t}\boldsymbol{\lambda}_{\omega_t}\}+\boldsymbol{\mu}'_{\omega_1}\mathbf{a}_0$$

where $\bar{\boldsymbol{\mu}}_{t+1|\omega_t}\equiv\sum_{\omega_{t+1}\in\mathcal{W}_{t+1|\omega_t}}p_{\omega_{t+1}|\omega_t}\boldsymbol{\mu}_{\omega_{t+1}}$ and $\bar{\boldsymbol{\mu}}_{T+1|\omega_T}\equiv\mathbf{0}$.

The optimization problem $\min_{\boldsymbol{\lambda}\geq\mathbf{0}}\max_{\mathbf{s}}L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ can then be decomposed as :

$$\sum_{t=1}^{T}\beta^{t-1}\sum_{\omega_t\in\mathcal{W}_t}p_{\omega_t}\min_{\boldsymbol{\lambda}_{\omega_t}\geq\mathbf{0}}\max_{\mathbf{s}_{\omega_t}}\{\Pi(\mathbf{s}_{\omega_t};\bar{\boldsymbol{\pi}}_{t|\omega_t}-\mathbf{D}'\boldsymbol{\mu}_{\omega_t}+\beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t})+\mathbf{s}'_{\omega_t}\boldsymbol{\lambda}_{\omega_t}\}.$$

Proposition B1 then allows showing result (*vi*), *i.e.* problem (D^d) is equivalent to the dual problem:

(DD^d) $V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\min_{\boldsymbol{\mu}}U(\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$

where :

$$U(\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\sum_{t=1}^{T}\beta^{t-1}\sum_{\omega_t\in\mathcal{W}_t}p_{\omega_t}\Pi^s(\bar{\boldsymbol{\pi}}_{t|\omega_t}-\mathbf{D}'\boldsymbol{\mu}_{\omega_t}+\beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t})+\boldsymbol{\mu}'_{\omega_1}\mathbf{a}_0.$$

Problem (D^d) being behaved, the solution set in $\boldsymbol{\mu}$ to problem (DD^d), *i.e.* :

$$\mathcal{M}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\min_{\boldsymbol{\mu}}U(\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$$

exists and the solution in **s** to problem (D^d) can be obtained with:

$$\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\mathbf{s}^s(\bar{\boldsymbol{\pi}}_{t|\omega_t}-\mathbf{D}'\boldsymbol{\mu}_{\omega_t}+\beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t})\text{ for }\omega_t\in\mathcal{W}_t\text{ and}$$

for any $\boldsymbol{\mu}\in\mathcal{M}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$.

The characterization of $\mathcal{M}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ in result (*v*) follows from:

$$\tfrac{\partial}{\partial\boldsymbol{\mu}_{\omega_1}}U(\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\mathbf{a}_0-\mathbf{D}\mathbf{s}^s(\bar{\boldsymbol{\pi}}_{1|\omega_1}-\mathbf{D}'\boldsymbol{\mu}_{\omega_1}+\beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{2|\omega_1})$$

and:

$$\frac{\partial}{\partial \boldsymbol{\mu}_{\omega_t}} U(\boldsymbol{\mu};\overline{\boldsymbol{\pi}},\mathbf{a}_0) = \beta^{t-1} \left\{ \begin{array}{c} \sum_{\omega_{t-1}\in\mathcal{W}_{t-1}} p_{\omega_{t-1}} p_{\omega_t|\omega_{t-1}} \mathbf{As}^s(\overline{\boldsymbol{\pi}}_{t-1|\omega_{t-1}} - \mathbf{D}'\boldsymbol{\mu}_{\omega_{t-1}} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t|\omega_{t-1}}) \\ -p_{\omega_t}\mathbf{Ds}^s(\overline{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t+1|\omega_t}) \end{array} \right\}.$$

for $\omega_t \in \mathcal{W}_t$ and $t = 2,...,T$. The specific structure of the ω_t and $\mathcal{W}_t$ terms imply that (*a*) there is a unique element ω_{t-1} of $\mathcal{W}_{t-1}$ such that $\omega_t \in \mathcal{W}_{t|\omega_{t-1}}$ and (*b*) for this element we have $p_{\omega_{t-1}} p_{\omega_t|\omega_{t-1}} = p_{\omega_t}$. This implies that:

$$\frac{\partial}{\partial \boldsymbol{\mu}_{\omega_t}} U(\boldsymbol{\mu};\overline{\boldsymbol{\pi}},\mathbf{a}_0) = \beta^{t-1} p_{\omega_t} \left\{ \begin{array}{l} \mathbf{As}^s(\overline{\boldsymbol{\pi}}_{t-1|\omega_{t-1}} - \mathbf{D}'\boldsymbol{\mu}_{\omega_{t-1}} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t|\omega_{t-1}}) \\ -\mathbf{Ds}^s(\overline{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t+1|\omega_t}) \end{array} \right\}$$

where ω_{t-1} is the a unique element of $\mathcal{W}_{t-1}$ such that $\omega_t \in \mathcal{W}_{t|\omega_{t-1}}$. If $\boldsymbol{\mu} \in \arg\min_{\boldsymbol{\mu}} U(\boldsymbol{\mu};\overline{\boldsymbol{\pi}},\mathbf{a}_0)$ then the first order conditions of the considered minimization problem provide:

$$\mathbf{Ds}^s(\overline{\boldsymbol{\pi}}_{1|\omega_1} - \mathbf{D}'\boldsymbol{\mu}_{\omega_1} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{2|\omega_1}) = \mathbf{a}_0$$

and:

$$\mathbf{Ds}^s(\overline{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t+1|\omega_t}) = \mathbf{As}^s(\overline{\boldsymbol{\pi}}_{t-1|\omega_{t-1}} - \mathbf{D}'\boldsymbol{\mu}_{\omega_{t-1}} + \beta\mathbf{A}'\overline{\boldsymbol{\mu}}_{t|\omega_{t-1}}).$$

for $\omega_t \in \mathcal{W}_{t|\omega_{t-1}}$, $\omega_{t-1} \in \mathcal{W}_{t-1}$ and $t = 2,...,T$. This yields result (*v*).

The continuous differentiability of $V^o_{\omega_1}$ in $\overline{\boldsymbol{\pi}}$ can be demonstrated following the approach employed for proving problem continuous differentiability of value function (M). From a technical viewpoint, problem (D^d) can be interpreted as a (very) large myopic problem. When applied to derivatives of $V^o_{\omega_1}$ Theorem 58 of Berkelaar *et al* (1997) indicates that $V^o_{\omega_t}$ is differentiable in $\overline{\boldsymbol{\pi}}$ if the solution in $\mathbf{s}$ to the problem defining $V^o_{\omega_t}$ is unique. The continuous differentiability of $V^o_{\omega_1}$ follows the uniqueness of $\mathbf{s}^o(\overline{\boldsymbol{\pi}},\mathbf{a}_0)$. Provided that

$$\frac{\partial}{\partial \overline{\boldsymbol{\pi}}} V^o_{\omega_1}(\overline{\boldsymbol{\pi}},\mathbf{a}_0) = \mathbf{s}^o(\overline{\boldsymbol{\pi}},\mathbf{a}_0)$$

and that $\mathbf{s}^o(\overline{\boldsymbol{\pi}},\mathbf{a}_0)$ is continuous in $\overline{\boldsymbol{\pi}}$ on $\mathbb{R}^{NW} \times \mathcal{A}$, $V^o_{\omega_1}$ is continuously differentiable in $\overline{\boldsymbol{\pi}}$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$.

Result (*vii*) is demonstrated in the proof of Proposition D3 which provides detailed results on these technical issues.

QED.

The next proposition establishes the properties of the value functions of problem (D^d) which are defined according to Bellman's dynamic programming principle.

Proposition D2. *Dynamic problem, dynamic programming approach*

Let consider the dynamic crop rotation problem defined in proposition D1. Let define the vector $\bar{\boldsymbol{\pi}}^{+}_{t|\omega_t}$ as $\bar{\boldsymbol{\pi}}^{+}_{t|\omega_t} \equiv (\bar{\boldsymbol{\pi}}_{t|\omega_t}, \bar{\boldsymbol{\pi}}^{+}_{(\omega_t)})$ where $\bar{\boldsymbol{\pi}}^{+}_{(\omega_t)} \equiv (\bar{\boldsymbol{\pi}}_{\tau|\omega_\tau} : \omega_\tau \in \mathcal{W}_\tau(\omega_{\tau-1}),\ \tau = t+1,...,T)$, with the convention $\bar{\boldsymbol{\pi}}^{+}_{1|\omega_1} = \bar{\boldsymbol{\pi}}$, and let define W_t^+ as $W_t^+ \equiv \dim \bar{\boldsymbol{\pi}}^{+}_{t|\omega_t}$.

(*i*) Let $\mathbf{a}_{t-1} \in \mathcal{A}$ denote the preceding crop acreage at date *t*. The value functions associated to problem (D^d) are recursively defined as:

$$V^o_{\omega_t}(\bar{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1}) \equiv \max_{\mathbf{s}_{\omega_t} \geq \mathbf{0}} \{\Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t}) + \beta \bar{V}^o_{t+1|\omega_t}(\bar{\boldsymbol{\pi}}^{+}_{(\omega_t)}, \mathbf{A}\mathbf{s}_{\omega_t}) \text{ s.t. } \mathbf{D}\mathbf{s}_{\omega_t} = \mathbf{a}_{t-1}\}$$

where :

$$\bar{V}^o_{t+1|\omega_t}(\bar{\boldsymbol{\pi}}^{+}_{(\omega_t)}, \mathbf{a}_t) \equiv \sum_{\omega_{t+1} \in \mathcal{W}_{t+1}(\omega_t)} p_{\omega_{t+1}|\omega_t} V^o_{\omega_{t+1}}(\bar{\boldsymbol{\pi}}^{+}_{t+1|\omega_{t+1}}, \mathbf{a}_t)$$

for $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$, with the convention $V^o_{\omega_{T+1}}(\bar{\boldsymbol{\pi}}^{+}_{(\omega_T)}, \mathbf{a}_T) \equiv 0$.

(*ii*) For any $(\bar{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1}) \in \mathbb{R}^{NW_t} \times \mathcal{A}$, $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$, the optimal choice of $\mathbf{s}_{\omega_t}$ in the maximization problem defining $V^o_{\omega_t}(\bar{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$ is unique:

$$\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_{t-1}) \equiv \arg\max_{\mathbf{s}_{\omega_t} \geq \mathbf{0}} \{\Pi(\mathbf{s}_{\omega_t}; \bar{\boldsymbol{\pi}}_{t|\omega_t}) + \beta \bar{V}^o_{\omega_{t+1}|\omega_t}(\bar{\boldsymbol{\pi}}^{+}_{(\omega_t)}, \mathbf{A}\mathbf{s}_{\omega_t}) \text{ s.t. } \mathbf{D}\mathbf{s}_{\omega_t} - \mathbf{a}_{t-1}\}.$$

(*iii*) For any $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$, the value function $V^o_{\omega_t} : \mathbb{R}^{NW_t} \times \mathcal{A} \to \mathbb{R}$ is:

(a) piecewise quadratic and continuous in $(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$, convex in $\overline{\boldsymbol{\pi}}$ and concave in $\mathbf{a}_{t-1}$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$,

(b) continuously differentiable in $\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$ with

$\frac{\partial}{\partial \overline{\boldsymbol{\pi}}_{\omega_t}} V^{o}_{\omega_t}(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1}) = \mathbf{s}^{o}_{\omega_t}(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$,

(c) continuously differentiable in $\mathbf{a}_{t-1}$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$. *I.e.* there exists a function $\boldsymbol{\mu}^{o}_{\omega_t} : \mathbb{R}^{NW_t} \times \mathcal{A} \to \mathbb{R}^{K}$, continuous in $(\overline{\boldsymbol{\pi}}, \mathbf{a}_{t-1})$ on $\mathbb{R}^{NW_t} \times \mathcal{A}$, such that

$\frac{\partial}{\partial \mathbf{a}_{t-1}} V^{o}_{\omega_t}(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1}) = \boldsymbol{\mu}^{o}_{\omega_t}(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$.

Proof. The recursive definition of the value functions in result (i) is an application of Bellman's dynamic programming principle. In the considered case this principle allows decomposing the large strictly quadratic programming problem (D^{d}) into smaller ones. This provides result (i). This also implies that the value functions functions $V^{o}_{\omega_t}$ for $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$ have the properties of the value function $V^{o}_{\omega_1}$ stated in Proposition D3.

QED.

The properties of the value functions $V^{o}_{\omega_t}$ in $(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$ for $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$ are identical to those of $V^{o}_{\omega_1}$ in $(\overline{\boldsymbol{\pi}}, \mathbf{a}_0)$ because these value functions are the solution function to "small" versions of problem (D^{d}).

$\mathbf{s}^{o}_{\omega_t}(\overline{\boldsymbol{\pi}}^{+}_{t|\omega_t}, \mathbf{a}_{t-1})$ is the optimal contingent acreage choice in sub-scenario ω_t with previous crop acreage $\mathbf{a}_{t-1}$. This acreage choice is equal to the optimal acreage choice in sub-scenario ω_t of problem (D^{d}) if and only if $\mathbf{a}_{t-1} = \mathbf{A}\mathbf{s}^{o}_{\omega_{t-1}}(\overline{\boldsymbol{\pi}}, \mathbf{a}_0)$ where ω_{t-1} is the sub-scenario from date

1 to date $t-1$ corresponding to sub-scenario ω_t. *I.e.* $\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}}^+_{t|\omega_t},\mathbf{a}_{t-1})=\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ if and only $\mathbf{a}_{t-1}$ is equal to the optimal previous acreage choice of problem (D^d) in the only sub-scenario happening at date $t-1$ satisfying $\omega_t \in \mathcal{W}_t(\omega_{t-1})$.

This shows that problem (D^d) isn't easy to solve neither by following a stochastic programming approach (Proposition D1), nor by following a dynamic programming approach (Proposition D2).

The next proposition establishes some results related to the derivatives of the value function $V^o_{\omega_1}$, and thus to the value functions $V^o_{\omega_t}$ for $\omega_t \in \mathcal{W}_t$ and $t=1,...,T$.

Proposition D3. *Dynamic problem, crop rotation Lagrange multipliers*.

Let consider the dynamic crop rotation problem, problem (D^d), defined in proposition D1 and its value function $V^o_{\omega_1}:\mathbb{R}^{NW}\times\mathcal{A}\rightarrow\mathbb{R}$. The Lagrangian problem associated to problem (D^d) is defined by:

(LD^d) $\min_{\boldsymbol{\mu}}\min_{\boldsymbol{\lambda}\geq\mathbf{0}}\max_{\mathbf{s}} L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$

where:

$$\begin{aligned}L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)\equiv\ &\Pi(\mathbf{s}_{\omega_1};\bar{\boldsymbol{\pi}}_{1|\omega_1})+\mathbf{s}'_{\omega_1}\boldsymbol{\lambda}_{\omega_1}+\boldsymbol{\mu}'_{\omega_1}(\mathbf{a}_0-\mathbf{D}\mathbf{s}_{\omega_1})\\&+\sum\nolimits_{t=2}^{T}\beta^{t-1}\sum\nolimits_{\omega_t\in\mathcal{W}_{t|\omega_{t-1}}}p_{\omega_t}\{\Pi(\mathbf{s}_{\omega_t};\bar{\boldsymbol{\pi}}_{t|\omega_t})+\mathbf{s}'_{\omega_t}\boldsymbol{\lambda}_{\omega_t}\}\\&+\sum\nolimits_{t=2}^{T}\beta^{t-1}\sum\nolimits_{\omega_{t-1}\in\mathcal{W}_{t-1}}p_{\omega_{t-1}}\sum\nolimits_{\omega_t\in\mathcal{W}_{t|\omega_{t-1}}}p_{\omega_t|\omega_{t-1}}\boldsymbol{\mu}'_{\omega_t}(\mathbf{A}\mathbf{s}_{\omega_{t-1}}-\mathbf{D}\mathbf{s}_{\omega_t}).\end{aligned}$$

Let define the set of solutions in $(\boldsymbol{\lambda},\boldsymbol{\mu})$ to problem (LD^d) by $\mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$.

There exists a *K*-dimensional function $\boldsymbol{\mu}^o_{\omega_t}:\mathbb{R}^{NW}\times\mathcal{A}\rightarrow\mathbb{R}^K$ such that:

(*a*) if $a_{n,0}>0$ then $(\boldsymbol{\mu},\boldsymbol{\lambda})\in\mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ implies $\mu_{n,\omega_1}=\mu^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ and:

$$\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\equiv\mu^o_{n,\omega_1}=\tfrac{\partial}{\partial a^+_{n,0}}V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\tfrac{\partial}{\partial a^+_{n,0}}V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\tfrac{\partial}{\partial a_{n,0}}V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0),$$

(b) if $a_{n,0} = 0$ then $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min_{(\boldsymbol{\mu},\boldsymbol{\lambda})}\{\mu_{n,\omega_1}$ s.t. $(\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ and:

$$\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a^+_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0),$$

(c) $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$

and:

(d) $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$.

Proof. The differentiability properties of the value functions $V^o_{\omega_1}$ in $\mathbf{a}_0$ are demonstrated by considering of the Lagrangian problem associated to problem (D^d) used in the proof of Proposition D1, *i.e.*:

(LD^d) $\min_{\boldsymbol{\mu}} \min_{\boldsymbol{\lambda}\geq\mathbf{0}} \max_{\mathbf{s}} L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0)$

where:

$$\begin{aligned} L(\mathbf{s},\boldsymbol{\lambda},\boldsymbol{\mu};\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv{}& \Pi(\mathbf{s}_{\omega_1};\bar{\boldsymbol{\pi}}_{1|\omega_1}) + \mathbf{s}'_{\omega_1}\boldsymbol{\lambda}_{\omega_1} + \boldsymbol{\mu}'_{\omega_1}(\mathbf{a}_0 - \mathbf{D}\mathbf{s}_{\omega_1}) \\ &+ \sum_{t=2}^{T} \beta^{t-1} \sum_{\omega_t \in \mathcal{W}_{t|\omega_{t-1}}} p_{\omega_t}\{\Pi(\mathbf{s}_{\omega_t};\bar{\boldsymbol{\pi}}_{t|\omega_t}) + \mathbf{s}'_{\omega_t}\boldsymbol{\lambda}_{\omega_t}\} \\ &+ \sum_{t=2}^{T} \beta^{t-1} \sum_{\omega_{t-1} \in \mathcal{W}_{t-1}} p_{\omega_{t-1}} \sum_{\omega_t \in \mathcal{W}_{t|\omega_{t-1}}} p_{\omega_t|\omega_{t-1}} \boldsymbol{\mu}'_{\omega_t}(\mathbf{A}\mathbf{s}_{\omega_{t-1}} - \mathbf{D}\mathbf{s}_{\omega_t}). \end{aligned}$$

We know that the solution in $\mathbf{s}$ to problem (LD^d) , $\mathbf{s}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$, is unique. Let define the set of solutions in $(\boldsymbol{\lambda},\boldsymbol{\mu})$ to problem (LD^d) by $\mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \subset \mathbb{R}^{NW}_+ \times \mathbb{R}^{KW}$. $Q^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ is characterized by the KKT conditions of problem (LD^d):

$$\begin{cases} \bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{h}_{\omega_t} + \boldsymbol{\lambda}_{\omega_t} - \mathbf{D}'\boldsymbol{\mu}_{\omega_t} + \beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t} - \mathbf{H}\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \mathbf{0} \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,...,T \\ \mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)'\boldsymbol{\lambda}_{\omega_t} = 0,\ \mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \geq \mathbf{0},\ \boldsymbol{\lambda}_{\omega_t} \geq \mathbf{0} \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,...,T \\ \mathbf{D}\mathbf{s}_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) - \mathbf{a}_0 = \mathbf{0} \\ \mathbf{D}\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) - \mathbf{A}\mathbf{s}^o_{\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \mathbf{0} \text{ for } \omega_t \in \mathcal{W}_t(\omega_{t-1}),\ \omega_{t-1} \in \mathcal{W}_{t-1} \text{ and } t = 2,...,T. \end{cases}$$

where $\bar{\boldsymbol{\mu}}_{t+1|\omega_t} \equiv \sum_{\omega_{t+1} \in \mathcal{W}_{t+1}(\omega_t)} p_{\omega_{t+1}|\omega_t}\boldsymbol{\mu}_{\omega_{t+1}}$ and $\bar{\boldsymbol{\mu}}_{T+1|\omega_T} = \mathbf{0}$ because $\boldsymbol{\mu}_{\omega_{T+1}} \equiv \mathbf{0}$. By these KKT conditions we know that the terms

$$\boldsymbol{\eta}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \mathbf{D}'\boldsymbol{\mu}_{\omega_t} - \boldsymbol{\lambda}_{\omega_t} - \beta\mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t} \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,...,T-1$$

and

$$\boldsymbol{\eta}^o_{\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \mathbf{D}'\boldsymbol{\mu}_{\omega_T} - \boldsymbol{\lambda}_{\omega_T} \text{ for } \omega_T \in \mathcal{W}_T$$

are uniquely defined. It suffices to observe that $\boldsymbol{\eta}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \bar{\boldsymbol{\pi}}_{t|\omega_t} - \mathbf{h} - \mathbf{H}\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ and that the terms $\mathbf{s}^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ are uniquely defined for $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$. Moreover, $\mathbf{s}^o_{\omega_t}$ being continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^N \times \mathcal{A}$, $\boldsymbol{\eta}^o_{\omega_t} : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}^N$ is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^N \times \mathcal{A}$ for $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T$.

When applied to $V^o_{\omega_1}$ for derivatives in $a_{n,0}$ Theorem 50 of Berkelaar *et al* (1997) yields:

$$\tfrac{\partial}{\partial a^+_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min_{(\boldsymbol{\mu},\boldsymbol{\lambda})}\{\mu_{n,\omega_1} \text{ s.t. } (\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$$

and:

$$\tfrac{\partial}{\partial a^-_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \max_{(\boldsymbol{\mu},\boldsymbol{\lambda})}\{\mu_{n,\omega_1} \text{ s.t. } (\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}.$$

We want to show that there exists a K-dimensional function $\boldsymbol{\mu}^o_{\omega_t} : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}^K$ such that:

(a) if $a_{n,0} > 0$ then $(\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ implies $\mu_{n,\omega_1} = \mu^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ and:

$$\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \mu^o_{n,\omega_1} = \tfrac{\partial}{\partial a^+_{n,0}} V^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a^+_{n,0}} V^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a_{n,0}} V^o_{\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0),$$

(b) if $a_{n,0} = 0$ then $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min_{(\boldsymbol{\mu},\boldsymbol{\lambda})}\{\mu_{n,\omega_1} \text{ s.t. } (\boldsymbol{\mu},\boldsymbol{\lambda}) \in \mathcal{D}^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ and:

$$\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a^+_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0),$$

(c) $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$

and:

(d) $\mu^o_{n,\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \tfrac{\partial}{\partial a_{n,0}} V^o_{\omega_1}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$.

Note that conditions (a)-(c) imply condition (d) and that condition (d) implies that $V^{o}_{\omega_1}$ is continuously differentiable in $\mathbf{a}_0$ on $\mathbb{R}^{NW} \times \mathcal{A}$.

We proceed by backward induction and by using the KKT condition equation system and, in particular:

$$\begin{cases} \mathbf{D}'\boldsymbol{\mu}_{\omega_T} - \boldsymbol{\lambda}_{\omega_T} = \boldsymbol{\eta}^{o}_{\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \text{ for } \omega_T \in \mathcal{W}_T \\ \mathbf{D}'\boldsymbol{\mu}_{\omega_t} - \boldsymbol{\lambda}_{\omega_t} - \beta \mathbf{A}'\bar{\boldsymbol{\mu}}_{t+1|\omega_t} = \boldsymbol{\eta}^{o}_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,\dots,T-1 \\ \mathbf{s}^{o}_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)'\boldsymbol{\lambda}_{\omega_t} = 0 \text{ and } \boldsymbol{\lambda}_{\omega_t} \geq \mathbf{0} \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,\dots,T \end{cases}$$

with $\mathbf{s}^{o}_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \geq \mathbf{0}$ or, equivalently:

$$\begin{cases} \mu_{n,\omega_T} - \lambda_{nk,\omega_T} = \eta^{o}_{nk,\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \text{ for } \omega_T \in \mathcal{W}_T \\ \mu_{n,\omega_t} - \lambda_{nk,\omega_t} - \beta \bar{\mu}_{k,t+1|\omega_t} = \eta^{o}_{nk,\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,\dots,T-1 \\ s^{o}_{nk,\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0)\lambda_{nk,\omega_t} = 0 \text{ and } \lambda_{nk,\omega_t} \geq 0 \text{ for } \omega_t \in \mathcal{W}_t \text{ and } t = 1,\dots,T \\ \text{for } (n,k) \in \mathcal{K} \times \mathcal{K}. \end{cases}$$

Let define the sets:

$$Q^{o}_{n,\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \left\{ (\mu_{n,\omega_T}, \boldsymbol{\lambda}_{(n),\omega_T}) \in \mathbb{R}^{K+1} : \left| \begin{array}{c} \mu_{n,\omega_T} \times \boldsymbol{\iota} - \boldsymbol{\lambda}_{(n),\omega_T} = \boldsymbol{\eta}^{o}_{\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \\ \boldsymbol{\lambda}'_{(n),\omega_T}\mathbf{s}^{o}_{(n),\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) = 0,\ \boldsymbol{\lambda}_{(n),\omega_t} \geq \mathbf{0} \end{array} \right. \right\}$$

and :

$$\mathcal{M}^{o}_{n,\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \{ \mu_{n,\omega_T} \in \mathbb{R} : (\boldsymbol{\mu}_{\omega_T}, \boldsymbol{\lambda}_{\omega_T}) \in Q_{\omega_T}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \}$$

for $n \in \mathcal{K}$ and $\omega_T \in \mathcal{W}_T$, and:

$$Q^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \left\{ (\mu_{n,\omega_t}, \boldsymbol{\lambda}_{(n),\omega_t}) \in \mathbb{R}^{K+1} : \left| \begin{array}{c} \mu_{n,\omega_t} \times \boldsymbol{\iota} - \boldsymbol{\lambda}_{(n),\omega_t} - \beta \bar{\boldsymbol{\mu}}_{t+1|\omega_t} = \boldsymbol{\eta}^{o}_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \\ \boldsymbol{\lambda}'_{(n),\omega_t}\mathbf{s}^{o}_{(n),\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) = 0,\ \boldsymbol{\lambda}_{(n),\omega_t} \geq \mathbf{0} \\ \bar{\boldsymbol{\mu}}_{t+1|\omega_t} \in \bar{\mathcal{M}}^{o}_{n,t+1|\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \end{array} \right. \right\},$$

$$\mathcal{M}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \equiv \{ \mu_{n,\omega_t} \in \mathbb{R} : (\boldsymbol{\mu}_{\omega_t}, \boldsymbol{\lambda}_{\omega_t}) \in Q_{\omega_t}(\bar{\boldsymbol{\pi}}, \mathbf{a}_0) \}$$

and:

$$\bar{\mathcal{M}}^o_{n,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \left\{ \mu_n \in \mathbb{R} : \left| \begin{array}{c} \mu_n = \sum_{\omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t)} p_{\omega_{t+1}|\omega_t}\mu_{n,\omega_{t+1}} \\ \mu_{n,\omega_{t+1}} \in \mathcal{M}^o_{n,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \text{ for } \omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t) \end{array} \right. \right\}$$

for $n\in\mathcal{K}$, $\omega_t \in \mathcal{W}_t$ and $t = 1,...,T-1$.

We want to show by backward recursion that for any $t = 1,...,T-1$, $\omega_{t-1}\in\mathcal{W}_{t-1}$, $\omega_t\in\mathcal{W}_t(\omega_{t-1})$ and $n\in\mathcal{K}$ the function $\mu^o_{n,\omega_t} : \mathbb{R}^{NW}\times\mathcal{A}\to\mathbb{R}$ defined by:

$$\mu^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \max\{\eta^o_{nk,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) + \beta\bar{\mu}^o_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) : k\in\mathcal{K}\}$$

satisfies:

($a.t$) $\mathcal{M}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \{\mu^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) > 0$ (or $a_{n,0} > 0$ if $t = 1$),

($b.t$) $\mu^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min\{\mu_{n,\omega_t} \in \mathcal{M}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = 0$ (or $a_{n,0} = 0$ if $t = 1$),

($c.t$) μ^o_{n,ω_t} is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW}\times\mathcal{A}$ for any $n\in\mathcal{K}$.

Note that conditions ($a.t$) and ($b.t$) imply that $\mu^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min\{\mu_{n,\omega_t} \in \mathcal{M}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ for any $\omega_t\in\mathcal{W}_t$ and $n\in\mathcal{K}$.

First, let assume that the function $\mu^o_{n,\omega_{t+1}} : \mathbb{R}^{NW}\times\mathcal{A}\to\mathbb{R}$ satisfies:

$(a.\,t+1)$ $\mathcal{M}^o_{n,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \{\mu^o_{n,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) > 0$,

$(b.\,t+1)$ $\mu^o_{n,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min\{\mu_{n,\omega_{t+1}} \in \mathcal{M}^o_{n,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = 0$,

$(c.\,t+1)$ $\mu^o_{n,\omega_{t+1}}$ is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW}\times\mathcal{A}$ for any $n\in\mathcal{K}$

for any $\omega_t\in\mathcal{W}_t$, $\omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t)$ and $n\in\mathcal{K}$.

We aim at showing that μ^o_{n,ω_t} satisfies conditions ($a.t$)-($c.t$) if conditions ($a.$ $t+1$)-($c.$ $t+1$) hold for $\mu^o_{n,\omega_{t+1}}$ for $\omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t)$ and $n\in\mathcal{K}$.

Let define the functions $\bar{\mu}^{o}_{\ell,\omega_{t+1}} : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}$:

$$\bar{\mu}^{o}_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \sum_{\omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t)} p_{\omega_{t+1}|\omega_t}\mu^{o}_{k,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$$

for $\omega_t \in \mathcal{W}_t$ and $k \in \mathcal{K}$. From the continuity of $\mu^{o}_{k,\omega_{t+1}}$ in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$ we know that $\bar{\mu}^{o}_{k,t+1|\omega_t} : \mathbb{R}^{NW} \times \mathcal{A} \to \mathbb{R}$ is also continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$. This implies, together with the continuity of $\boldsymbol{\eta}^{o}_{\omega_t}$ in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW} \times \mathcal{A}$, that condition (*c.t*) necessarily holds.

With $\mu^{o}_{k,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \equiv \min\{\mu_{k,\omega_{t+1}} \in \mathcal{M}^{o}_{k,\omega_{t+1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ and $p_{\omega_{t+1}|\omega_t} > 0$ for $\omega_{t+1} \in \mathcal{W}_{t+1}(\omega_t)$ we have $\bar{\mu}^{o}_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = \min\{\bar{\mu}_{k,t+1|\omega_t} \in \bar{\mathcal{M}}^{o}_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$. Note also that, for any $(n,\omega_t) \in \mathcal{K}\times\mathcal{W}_t$, there exists $(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}) \in \mathcal{Q}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ with $\lambda_{nk,\omega_t} = 0$ for some $k \in \mathcal{K}$. Two cases may occur, depending on whether $\boldsymbol{\iota}'\mathbf{s}^{o}_{n,\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) > 0$ or $\boldsymbol{\iota}'\mathbf{s}^{o}_{n,\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = 0$. If $\boldsymbol{\iota}'\mathbf{s}^{o}_{n,\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) > 0$ then there necessarily exists $k \in \mathcal{K}$ such that $s^{o}_{nk,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) > 0$ and we necessarily have $\lambda_{nk,\omega_t} = 0$ if $(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}) \in \mathcal{Q}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$. If $\boldsymbol{\iota}'\mathbf{s}^{o}_{n,\omega_{t-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) = 0$ and $(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}) \in \mathcal{Q}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ then we necessarily have $\lambda_{nk,\omega_t} = 0$ for $k \in \arg\max\{\eta^{o}_{n\ell,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) + \beta\bar{\mu}_{\ell,t+1|\omega_t} : \ell \in \mathcal{K}\}$ for any $\bar{\mu}_{k,t+1|\omega_t} \in \bar{\mathcal{M}}^{o}_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ as long as:

$$\mu_{n,\omega_t} = \min_{\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}\geq\mathbf{0}}\{\mu_{n,\omega_t} \text{ s.t. } (\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}) \in \mathcal{Q}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}.$$

With:

$$(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t}) \in \mathcal{Q}^{o}_{n,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \iff \begin{cases} \mu_{n,\omega_t} = \eta^{o}_{nk,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) + \lambda_{nk,\omega_t} + \beta\bar{\mu}_{k,t+1|\omega_t} \quad \text{for } k \in \mathcal{K} \\ \bar{\mu}_{k,t+1|\omega_t} \in \bar{\mathcal{M}}^{o}_{k,t+1|\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0) \\ s^{o}_{nk,\omega_t}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\lambda_{nk,\omega_t} = 0 \text{ and } \lambda_{nk,\omega_t} \geq 0 \quad \text{for } k \in \mathcal{K} \end{cases}$$

the results obtained above imply that:

$$\min_{\mu_{n,\omega_t}}\{\mu_{n,\omega_t}\in\mathcal{M}^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$$
$$=$$
$$\min_{(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t})}\{\mu_{n,\omega_t}:(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t})\in Q^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$$
$$=$$
$$\max_{k\in\mathcal{K}}\left\{\min_{(\lambda_{nk,\omega_t},\overline{\mu}_{k,t+1|\omega_t})}\left\{\begin{array}{c}\eta^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)+\lambda_{nk,\omega_t}+\beta\overline{\mu}_{k,t+1|\omega_t}\\ \text{s.t.}\\ \overline{\mu}_{k,t+1|\omega_t}\in\overline{\mathcal{M}}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0),\ \lambda_{nk,\omega_t}s^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)=0,\ \lambda_{nk,\omega_t}\geq 0\end{array}\right\}\right\}.$$

and, as a result that :

$$\min_{\mu_{n,\omega_t}}\{\mu_{n,\omega_t}\in\mathcal{M}^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$$
$$=$$
$$\max_{k\in\mathcal{K}}\left\{\min_{\lambda_{nk,\omega_t}}\left\{\begin{array}{c}\eta^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)+\lambda_{nk,\omega_t}+\beta\overline{\mu}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\\ \text{s.t.}\\ \lambda_{nk,\omega_t}s^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)=0,\ \lambda_{nk,\omega_t}\geq 0\end{array}\right\}\right\}$$
$$=$$
$$\max\{\eta^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)+\beta\overline{\mu}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0):k\in\mathcal{K}\}.$$

Provided that $\mu^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\equiv\max\{\eta^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)+\beta\overline{\mu}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0):k\in\mathcal{K}\}$, this provides condition $(b.\,t+1)$.

If $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{t-1}}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)>0$ (or $a_{n,0}>0$ if $t=1$), then there necessarily exists $k\in\mathcal{K}$ such that $s^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)>0$ and $\lambda_{nk,\omega_t}=0$ if $(\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t})\in Q^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)$. This in turn implies that $\boldsymbol{\iota}'\mathbf{s}^o_{k,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)>0$ and, by condition $(a.\,t+1)$, that $\mathcal{M}^o_{n,\omega_{t+1}}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)=\{\mu^o_{n,\omega_{t+1}}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$ for any $\omega_{t+1}\in\mathcal{W}_{t+1}(\omega_t)$ and, thus that $\overline{\mathcal{M}}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)=\{\overline{\mu}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$. Finally, we obtain that:

$$\mu_{n,\omega_t}=\eta^o_{nk,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)+\beta\overline{\mu}^o_{k,t+1|\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\ \text{for any}\ (\mu_{n,\omega_t},\boldsymbol{\lambda}_{(n),\omega_t})\in Q^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0).$$

As a result, we obtain that the optimal value μ_{n,ω_t} is uniquely defined, *i.e.* that $\mathcal{M}^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)=\{\mu^o_{n,\omega_t}(\overline{\boldsymbol{\pi}},\mathbf{a}_0)\}$ or that condition (*a.t*) holds.

Second, for completing the proof we need to show that conditions (*a.t*)-(*c.t*) hold for μ^o_{n,ω_t} for $t=T$, $\omega_{T-1}\in\mathcal{W}_{T-1}$, $\omega_T\in\mathcal{W}_T(\omega_{T-1})$ and $n\in\mathcal{K}$. Of course in this case we have $\bar{\mu}^o_{k,T+1|\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\bar{\mu}_{k,T+1|\omega_T}=\mu_{k,\omega_{T+1}}=0$.

The conditions defining $Q^o_{\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ correspond to the first order condition in $(\boldsymbol{\lambda}_{\omega_T},\boldsymbol{\mu}_{\omega_T})$ of a myopic problem, *i.e*. to that of:

$$\max_{\mathbf{s}_{\omega_T}}\{\Pi(\mathbf{s}_{\omega_T};\bar{\boldsymbol{\pi}}_{T|\omega_T})\text{ s.t. }\mathbf{D}\mathbf{s}_{\omega_T}=\mathbf{A}\mathbf{s}^o_{\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=0\text{ and }\mathbf{s}_{\omega_T}\geq\mathbf{0}\}$$

for $\omega_T\in\mathcal{W}_T(\omega_{T-1})$ and $\omega_{T-1}\in\mathcal{W}_{T-1}$. Hence, from Proposition C1 we know that for any $\omega_{T-1}\in\mathcal{W}_{T-1}$ and $\omega_T\in\mathcal{W}_T(\omega_{T-1})$ the function $\mu^o_{n,\omega_{t+1}}:\mathbb{R}^{NW}\times\mathcal{A}\to\mathbb{R}$ defined by:

$$\mu^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\equiv\max\{\eta^o_{nk,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0):k\in\mathcal{K}\}$$

satisfies:

(*a.T*) $\mathcal{M}^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\{\mu^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)>0$,

(*b.T*) $\mu^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=\min\{\mu\in\mathcal{M}^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\}$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=0$,

(*c.T*) μ^o_{n,ω_T} is continuous in $(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ on $\mathbb{R}^{NW}\times\mathcal{A}$ for any $n\in\mathcal{K}$.

Note that $\mathcal{M}^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ is not necessarily equal to $[\mu^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0),+\infty)$ if $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=0$ as in Proposition C1 because $\boldsymbol{\iota}'\mathbf{s}^o_{n,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)=0$ cannot hold if μ_{n,ω_T} is sufficiently large. Indeed, $(\boldsymbol{\mu}_{\omega_{T-1}},\boldsymbol{\lambda}_{\omega_{T-1}})\in Q^o_{\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ and $\mu_{n,\omega_T}\in\mathcal{M}^o_{n,\omega_T}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ for any $\omega_T\in\mathcal{W}_T(\omega_{T-1})$ imply:

$$\mu_{\ell,\omega_{T-1}}-\eta^o_{\ell n,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)\geq\beta\bar{\mu}_{n,T|\omega_{T-1}}=\sum_{\omega_T\in\mathcal{W}_T(\omega_{T-1})}p_{\omega_T|\omega_{T-1}}\mu_{n,\omega_T}\quad\text{for }\ell\in\mathcal{K}$$

while, as will be shown below, $\mu_{\ell,\omega_{T-1}}$ may be bounded from above (*e.g.*, $\mathcal{M}^o_{\ell,\omega_{T-1}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)$ is a singleton if $\boldsymbol{\iota}'\mathbf{s}^o_{\ell,\omega_{T-2}}(\bar{\boldsymbol{\pi}},\mathbf{a}_0)>0$).

QED.

Appendix E. Some useful results

This Appendix collects definitions and results related to nonlinear programming which are use in Appendices B-D.

The results collected in section E1 consider sensitivity analysis for general nonlinear programming problems. The results given in section E2 are related to quadratic multiparametric quadratic programming. They are mainly due to Bemporad *et al* (2002) and Berkelaar *et al* (1997).

E.1. Nonlinear programming sensitivity analysis

This section considers general nonlinear programming problems and sensitivity analysis of their solution functions. The results and definitions given in this section reproduce (with slight formal modifications) those collected in Fiacco and Kyparisis (1985). The Theorems are numbered as in Fiacco and Kyparisis (1985) and they cite references provided in this article.

Most of these results are well known. They consider the characterization of the solutions to nonlinear programming problems or sensitivity analysis for these solutions (*e.g.* results are known implicit function or envelope properties).

The results published in Jittorntrum (1984) are less frequently considered. These results are referred to as Jittorntrum (1978, 1981) in Fiacco and Kyparisis (1985).

E1.1. Necessary and sufficient conditions for local minima

Let consider the nonlinear programming problem

$$\text{(P)} \qquad \min_{\mathbf{x}}\{ f(\mathbf{x}) \text{ s.t. } g_i(\mathbf{x}) \geq 0 \text{ for } i=1,\ldots,m \text{ and } h_j(\mathbf{x})=0 \text{ for } j=1,\ldots,p\}$$

assuming that the functions f, $\mathbf{g}$ and $\mathbf{h}$ are twice continuous differentiable in $\mathbf{x}$ in a neighborhood of $\mathbf{x}^o$. The Lagrangian problem associated to problem (P) is given by

$$L(\mathbf{x},\mathbf{u},\mathbf{w}) \equiv f(\mathbf{x}) - \sum_{i=1}^{m} u_i g_i(\mathbf{x}) + \sum_{j=1}^{p} w_j h_j(\mathbf{x}) = f(\mathbf{x}) - \mathbf{u}'\mathbf{g}(\mathbf{x}) + \mathbf{w}'\mathbf{h}(\mathbf{x}) .$$

Definition. *Karush-Kuhn-Tucker* (KTT) *conditions*. The KKT conditions holds at $\mathbf{x}^o$ for problem (P) if there exists Lagrange multipliers $\mathbf{u}^o$ and $\mathbf{w}^o$ such that:

$$\tfrac{\partial}{\partial \mathbf{x}} L(\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o) = \mathbf{0}$$
$$u_i^o g_i(\mathbf{x}^o) = 0,\ u_i^o \geq 0 \text{ and } g_i(\mathbf{x}^o) \geq 0 \text{ for } i = 1,\ldots,m$$
$$h_j(\mathbf{x}^o) = 0 \text{ for } j = 1,\ldots,p$$

Theorem 2.1. *Necessary first order conditions for a local minimum*. Karush (1939) and Kuhn and Tucker (1951).

Suppose that $\mathbf{x}^o$ is a local minimum of problem (P) and an appropriate constraint qualifications hold at $\mathbf{x}^o$.

Then, the (KKT) conditions hold at $\mathbf{x}^o$ for problem (P). Conditions (LI) are appropriate constraint qualification conditions.

Definition. *Binding inequality constraint set*. $\mathcal{B}(\mathbf{x}^o) \equiv \{i = 1,\ldots,m : g_i^o(\mathbf{x}^o) = 0\}$

Definition. *Linear Independence conditions* (LI). Conditions (LI) hold at $\mathbf{x}^o$ for problem (P) if the vectors $\tfrac{\partial}{\partial \mathbf{x}} g_i(\mathbf{x}^o)$ for $i \in \mathcal{B}(\mathbf{x}^o)$ and $\tfrac{\partial}{\partial \mathbf{x}} h_j(\mathbf{x}^o)$ for $j = 1,\ldots,p$ are linearly independent.

If (LI) holds at $\mathbf{x}^o$ then $\mathbf{x}^o$ is said to be a regular point of problem (P).

Definition. *Second order necessary conditions* (SON). The second order necessary conditions hold for problem (P) at $\mathbf{x}^o$ with $\mathbf{u}^o$ and $\mathbf{w}^o$ if

$$\mathbf{z}' \tfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{x}'} L(\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)\mathbf{z} \geq 0$$

for all $\mathbf{z}$ such that:

$$\tfrac{\partial}{\partial \mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} \geq 0 \text{ if } i \in \mathcal{B}(\mathbf{x}^o)$$
$$\tfrac{\partial}{\partial \mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} = 0 \text{ if } i = 1, \ldots, m \text{ and } i \notin \mathcal{B}(\mathbf{x}^o)$$
$$\tfrac{\partial}{\partial \mathbf{x}'} h_j(\mathbf{x}^o)\mathbf{z} = 0 \text{ for } j = 1, \ldots, p$$

Theorem 2.2. *Second order necessary conditions* (SON), Fiacco and McCormick (1969) and McCormick (1976).

Suppose that $\mathbf{x}^o$ is a local minimum of problem (P) and that conditions (LI) for problem (P) hold at $\mathbf{x}^o$.

Then the conditions (KKT) and (SON) for problem (P) hold at $\mathbf{x}^o$ with associated unique Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$.

Definition. *Second order sufficient conditions for a strict local minimum* (SOS). The second order sufficient conditions for a strict local minimum hold for problem (P) at $\mathbf{x}^o$ with $\mathbf{u}^o$ and $\mathbf{w}^o$ if:

$$\mathbf{z}' \tfrac{\partial^2}{\partial \mathbf{x} \partial \mathbf{x}'} L(\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)\mathbf{z} > 0$$

for all $\mathbf{z} \neq \mathbf{0}$ such that:

$$\tfrac{\partial}{\partial \mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} \geq 0 \text{ for } i = 1, \ldots, m \text{ such that } i \in \mathcal{B}(\mathbf{x}^o)$$
$$\tfrac{\partial}{\partial \mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} = 0 \text{ for } i = 1, \ldots, m \text{ such that } u_i^o > 0$$
$$\tfrac{\partial}{\partial \mathbf{x}'} h_j(\mathbf{x}^o)\mathbf{z} = 0 \text{ for } j = 1, \ldots, p$$

Theorem 2.3. Pennisi (1953) and Fiacco and McCormick (1968).

Suppose conditions (KKT) hold at $\mathbf{x}^o$ with some Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$ and that conditions (SOS) also hold.

Then $\mathbf{x}^o$ is a strict local minimum of problem (P).

E1.2. Sensitivity analysis and implicit function properties

Let consider the perturbed nonlinear programming problem :

$$P(\boldsymbol{\varepsilon}) \quad \min_{\mathbf{x}}\{ f(\mathbf{x},\boldsymbol{\varepsilon}) \text{ s.t. } g_i(\mathbf{x},\boldsymbol{\varepsilon}) \geq 0 \text{ for } i=1,\ldots,m \text{ and } h_j(\mathbf{x},\boldsymbol{\varepsilon}) = 0 \text{ for } j=1,\ldots,p\}$$

where $\boldsymbol{\varepsilon}$ is a perturbation parameter assuming that the functions *f*, **g**, **h,** $\frac{\partial}{\partial \mathbf{x}} f$, $\frac{\partial}{\partial \mathbf{x}}\mathbf{g}$ and $\frac{\partial}{\partial \mathbf{x}}\mathbf{h}$ are continuous differentiable in $(\mathbf{x},\boldsymbol{\varepsilon})$ in a neighborhood of $(\mathbf{x}^o,\boldsymbol{\varepsilon}^o)$. The Lagrangian function associated with problem with problem $\mathrm{P}(\boldsymbol{\varepsilon})$ is given by

$$L(\mathbf{x},\mathbf{u},\mathbf{w},\boldsymbol{\varepsilon}) = f(\mathbf{x},\boldsymbol{\varepsilon}) - \mathbf{u}'\mathbf{g}(\mathbf{x},\boldsymbol{\varepsilon}) + \mathbf{w}'\mathbf{h}(\mathbf{x},\boldsymbol{\varepsilon}).$$

All results and definitions of Section 1 apply to problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$.

Definition. *Strict complementary slackness conditions* (SCS). The strict complementary slackness conditions hold for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ at $\mathbf{x}^o$ with respect to $\mathbf{u}^o$ if:

$$u_i^o > 0 \text{ for } i=1,\ldots,m \text{ such that } g_i(\mathbf{x}^o,\boldsymbol{\varepsilon}^o) = 0.$$

Theorem 3.1. Fiacco (1976).

Suppose that conditions (KKT), (SOS) and (LI) of problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ hold at $\mathbf{x}^o$ with associate Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$ and that the conditions (SCS) also hold.

Then,

(a) $\mathbf{x}^o$ is an isolated local minimum of $P(\boldsymbol{\varepsilon}^o)$ and the Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$ are unique;

(b) for $\boldsymbol{\varepsilon}$ in a neighborhood of $\boldsymbol{\varepsilon}^o$, there exists a continuously differentiable vector function $\mathbf{y}(\boldsymbol{\varepsilon}) \equiv \left(\mathbf{x}(\boldsymbol{\varepsilon}), \mathbf{u}(\boldsymbol{\varepsilon}), \mathbf{w}(\boldsymbol{\varepsilon})\right)$ satisfying conditions (KKT) and (SOS) of problem $P(\boldsymbol{\varepsilon})$ such that $\mathbf{y}(\boldsymbol{\varepsilon}^o) \equiv (\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)$, and, hence, $\mathbf{x}(\boldsymbol{\varepsilon})$ is a locally unique local minimum for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ with associated Lagrange multiplier vectors $\mathbf{u}(\boldsymbol{\varepsilon})$ and $\mathbf{w}(\boldsymbol{\varepsilon})$;

(c) the (LI) and (SCS) conditions hold at $\mathbf{x}(\boldsymbol{\varepsilon})$ for $\boldsymbol{\varepsilon}$ in a neighborhood of $\boldsymbol{\varepsilon}^o$.

The derivatives of $\mathbf{y}(\boldsymbol{\varepsilon})$ in $\boldsymbol{\varepsilon}$ in a neighborhood of $\boldsymbol{\varepsilon}^o$ can be calculated by using the conditions (KKT) of problem $P(\boldsymbol{\varepsilon})$ at $\mathbf{y}(\boldsymbol{\varepsilon})$:

$$\begin{gathered} \tfrac{\partial}{\partial \mathbf{x}'} L[\mathbf{x}(\boldsymbol{\varepsilon}), \mathbf{u}(\boldsymbol{\varepsilon}), \mathbf{w}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] = \mathbf{0} \\ u_i(\boldsymbol{\varepsilon}) g_i[\mathbf{x}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] = 0 \text{ for } i = 1, \dots, m \\ h_j[\mathbf{x}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] = 0 \text{ for } j = 1, \dots, p \end{gathered}$$

where:

$$\tfrac{\partial}{\partial \mathbf{x}} L[\mathbf{x}(\boldsymbol{\varepsilon}), \mathbf{u}(\boldsymbol{\varepsilon}), \mathbf{w}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] = \tfrac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}, \mathbf{w}, \boldsymbol{\varepsilon}) - \sum_{i=1}^{m} u_i(\boldsymbol{\varepsilon}) \tfrac{\partial}{\partial \mathbf{x}} g_i(\mathbf{x}, \boldsymbol{\varepsilon}) + \sum_{j=1}^{p} w_j(\boldsymbol{\varepsilon}) \tfrac{\partial}{\partial \mathbf{x}} h_j(\mathbf{x}, \boldsymbol{\varepsilon})$$

The assumptions of Theorem 3.1. imply that the Jacobian, $\mathbf{M}(\boldsymbol{\varepsilon})$, with respect to $\mathbf{y} \equiv (\mathbf{x}, \mathbf{u}, \mathbf{w})$ of this system of KKT conditions is nonsingular. As a result:

$$\mathbf{M}(\boldsymbol{\varepsilon}) \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{y}(\boldsymbol{\varepsilon}) = -\mathbf{Q}(\boldsymbol{\varepsilon}) \text{ and } \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{y}(\boldsymbol{\varepsilon}) = -\mathbf{M}(\boldsymbol{\varepsilon})^{-1} \mathbf{Q}(\boldsymbol{\varepsilon})$$

where $\mathbf{Q}(\boldsymbol{\varepsilon})$ is the Jacobian with respect to $\boldsymbol{\varepsilon}$ of this system of KKT conditions.

At $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^o$ we have:

$$\mathbf{M}(\boldsymbol{\varepsilon}^o)\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{y}(\boldsymbol{\varepsilon}^o)=\mathbf{M}(\boldsymbol{\varepsilon}^o)\begin{bmatrix}\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}(\boldsymbol{\varepsilon}^o)\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{u}(\boldsymbol{\varepsilon}^o)\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{w}(\boldsymbol{\varepsilon}^o)\end{bmatrix}=-\mathbf{Q}(\boldsymbol{\varepsilon}^o)$$

where :

$$\mathbf{M}\equiv\begin{bmatrix}\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}L & \tfrac{\partial}{\partial\mathbf{x}}g_1 & \cdots & \tfrac{\partial}{\partial\mathbf{x}}g_m & \tfrac{\partial}{\partial\mathbf{x}}h_1 & \cdots & \tfrac{\partial}{\partial\mathbf{x}}h_p\\ u_1\tfrac{\partial}{\partial\mathbf{x}'}g_1 & g_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ u_m\tfrac{\partial}{\partial\mathbf{x}'}g_m & \mathbf{0} & \mathbf{0} & g_m & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \tfrac{\partial}{\partial\mathbf{x}'}h_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \vdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \tfrac{\partial}{\partial\mathbf{x}'}h_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\quad\text{and}\quad\mathbf{Q}\equiv\begin{bmatrix}\tfrac{\partial^2}{\partial\mathbf{x}\partial\boldsymbol{\varepsilon}'}L\\ u_1\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}g_1\\ \vdots\\ u_m\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}g_m\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}h_1\\ \vdots\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}h_p\end{bmatrix}$$

i.e.:

$$\mathbf{M}\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{y}=\mathbf{M}\begin{bmatrix}\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{u}\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{w}\end{bmatrix}=\begin{bmatrix}\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}L\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}+\sum_{i=1}^m\tfrac{\partial}{\partial\mathbf{x}}g_i\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}u_i+\sum_{j=1}^p\tfrac{\partial}{\partial\mathbf{x}}h_j\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}w_j\\ u_1\tfrac{\partial}{\partial\mathbf{x}'}g_1\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}+g_1\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}u_1\\ \vdots\\ u_m\tfrac{\partial}{\partial\mathbf{x}'}g_m\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}+g_m\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}u_m\\ \tfrac{\partial}{\partial\mathbf{x}'}h_1\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}\\ \vdots\\ \tfrac{\partial}{\partial\mathbf{x}'}h_p\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}\mathbf{x}\end{bmatrix}=-\mathbf{Q}=-\begin{bmatrix}\tfrac{\partial^2}{\partial\mathbf{x}\partial\boldsymbol{\varepsilon}'}L\\ u_1\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}g_1\\ \vdots\\ u_m\tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}g_m\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}h_1\\ \vdots\\ \tfrac{\partial}{\partial\boldsymbol{\varepsilon}'}h_p\end{bmatrix}$$

and :

$$\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}L=\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}f-\sum_{i=1}^m u_i\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}g_i+\sum_{j=1}^p w_j\tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'}h_j$$

are evaluated at $(\mathbf{x}^o,\mathbf{u}^o,\mathbf{w}^o,\boldsymbol{\varepsilon}^o)$.

Theorem 3.2. McCormick (1979).

Suppose that conditions (KKT) and (SON) for problem $P(\boldsymbol{\varepsilon}^o)$ hold at $\mathbf{x}^o$ with associated Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$.

Then the Jacobian matrix $\mathbf{M}(\boldsymbol{\varepsilon}^o)$ is invertible if and only if conditions (SOS), (LI) and (SCS) for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ also hold at $\mathbf{x}^o$ with $\mathbf{u}^o$ and $\mathbf{w}^o$.

Definition. *Strong Second order sufficient conditions for a strict local minimum* (SSOS). The strong second order sufficient conditions for a strict local minimum hold for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ at $\mathbf{x}^o$ with $\mathbf{u}^o$ and $\mathbf{w}^o$ if:

$$\mathbf{z}' \tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'} L(\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)\mathbf{z} > 0$$

for all $\mathbf{z} \neq \mathbf{0}$ such that:

$$\tfrac{\partial}{\partial\mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} = 0 \text{ for } i = 1, ..., m \text{ such that } u_i^o > 0$$
$$\tfrac{\partial}{\partial\mathbf{x}'} h_j(\mathbf{x}^o)\mathbf{z} = 0 \text{ for } j = 1, ..., p$$

These conditions allow obtaining sensitivity analysis results without condition (SCS). Note that conditions (SOS) and (SSOS) differ by the conditions they place on **z**. The condition:

$$\tfrac{\partial}{\partial\mathbf{x}'} g_i(\mathbf{x}^o)\mathbf{z} \geq 0 \text{ for } i = 1, ..., m \text{ such that } i \in \mathcal{B}(\mathbf{x}^o)$$

in conditions (SOS) is omitted in conditions (SSOS). *I.e.* $\mathbf{z}' \tfrac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}'} L(\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)\mathbf{z} > 0$ must hold for set of **z** larger in conditions (SSOS) than in conditions (SOS).

Theorem 4.1. Jittortrum (1978, 1981) and Robinson (1980).

Suppose that conditions (KKT), (LI) and (SSOS) for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ hold at $\mathbf{x}^o$ with associated Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$.

Then,

(a) $\mathbf{x}^o$ is an isolated local minimum of problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$ and the Lagrange multiplier vectors $\mathbf{u}^o$ and $\mathbf{w}^o$ are unique;

(b) for $\boldsymbol{\varepsilon}$ in a neighborhood of $\boldsymbol{\varepsilon}^o$, there exists a continuous vector function $\mathbf{y}(\boldsymbol{\varepsilon}) \equiv (\mathbf{x}(\boldsymbol{\varepsilon}), \mathbf{u}(\boldsymbol{\varepsilon}), \mathbf{w}(\boldsymbol{\varepsilon}))$ satisfying conditions (KKT) and (SSOS) for problem $P(\boldsymbol{\varepsilon})$ such that $\mathbf{y}(\boldsymbol{\varepsilon}^o) \equiv (\mathbf{x}^o, \mathbf{u}^o, \mathbf{w}^o)$, and, hence, $\mathbf{x}(\boldsymbol{\varepsilon})$ is a locally unique local minimum of problem $P(\boldsymbol{\varepsilon})$ with associated unique Lagrange multiplier vectors $\mathbf{u}(\boldsymbol{\varepsilon})$ and $\mathbf{w}(\boldsymbol{\varepsilon})$;

(c) conditions (LI) hold at $\mathbf{x}(\boldsymbol{\varepsilon})$ for $\boldsymbol{\varepsilon}$ in a neighborhood of $\boldsymbol{\varepsilon}^o$;

(d) there exist $t > 0$ and $d > 0$ such that for all $\boldsymbol{\varepsilon}$ with $\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^o\| < d$, it follows that $\|\mathbf{y}(\boldsymbol{\varepsilon}) - \mathbf{y}(\boldsymbol{\varepsilon}^o)\| < t \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^o\|$.

The results collected in Theorem 4.1 (and in Theorem 7.3) were published in Jittorntrum (1984). Basically, Jittorntrum (1984) shows that the constraint qualification condition (SCS) condition is not necessary for obtaining implicit function (Theorem 4.1) and envelope (Theorem 7.3) properties as long as the considered problem has a unique solution is **s** and the constraint qualification condition (LI) holds at the optimum. Condition (LI) ensures that the solution to the dual problem in the Lagrange multipliers of the considered constraints is unique.

E1.3. Differentiability of the optimal value function and envelope properties

Definition. *Local optimal value function*. A local optimal value function of problem $P(\boldsymbol{\varepsilon})$ is defined as $f_\ell^o(\boldsymbol{\varepsilon}) \equiv f[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]$ where $\mathbf{x}^o(\boldsymbol{\varepsilon})$ is an isolated local minimum of problem $P(\boldsymbol{\varepsilon})$.

Theorem 7.2. Armacost and Fiacco (1978) and Fiacco (1980).

Suppose that the conditions (KKT), (SOSC), (LI) and (SCS) hold at $\mathbf{x}^o$ for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$.

Then, in a neighborhood of $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^o$, the local optimal value function f_ℓ^o is twice continuously differentiable and:

$$\text{(a)} \quad f_\ell^o(\boldsymbol{\varepsilon}) = L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}];$$

$$\text{(b)} \quad \begin{aligned} \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} f_\ell^o(\boldsymbol{\varepsilon}) &= \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] \\ &= \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} f[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] - \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{g}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \mathbf{u}^o(\boldsymbol{\varepsilon}) + \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{h}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \mathbf{w}^o(\boldsymbol{\varepsilon}) \end{aligned}$$

$$\text{(c)} \quad \begin{aligned} \tfrac{\partial^2}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}'} f_\ell^o(\boldsymbol{\varepsilon}) &= \tfrac{\partial^2}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}'} L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] - \mathbf{Q}^o(\boldsymbol{\varepsilon})' \mathbf{M}^o(\boldsymbol{\varepsilon})^{-1} \mathbf{Q}^o(\boldsymbol{\varepsilon}) \\ &= \tfrac{\partial^2}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}'} L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] + \tfrac{\partial^2}{\partial \boldsymbol{\varepsilon} \partial \mathbf{x}'} L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] \tfrac{\partial}{\partial \boldsymbol{\varepsilon}'} \mathbf{x}^o(\boldsymbol{\varepsilon}) \\ &\quad - \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{g}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \tfrac{\partial}{\partial \boldsymbol{\varepsilon}'} \mathbf{u}^o(\boldsymbol{\varepsilon}) + \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{h}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \tfrac{\partial}{\partial \boldsymbol{\varepsilon}'} \mathbf{w}^o(\boldsymbol{\varepsilon}) \end{aligned}$$

Theorem 7.3. Jittorntrum (1978, 1981).

Suppose that the conditions (KKT), (LI) and (SSOSC) hold at $\mathbf{x}^o$ for problem $\mathrm{P}(\boldsymbol{\varepsilon}^o)$.

Then, in a neighborhood of $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^o$, the local optimal value function f_ℓ^o is continuously differentiable with:

(a) $f_\ell^o(\boldsymbol{\varepsilon}) = L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]$,

(b) $\tfrac{\partial}{\partial \boldsymbol{\varepsilon}} f_\ell^o(\boldsymbol{\varepsilon}) = \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} L[\mathbf{x}^o(\boldsymbol{\varepsilon}), \mathbf{u}^o(\boldsymbol{\varepsilon}), \mathbf{w}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]$,

(c) $\tfrac{\partial}{\partial \boldsymbol{\varepsilon}} f_\ell^o(\boldsymbol{\varepsilon}) = \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} f[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}] - \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{g}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \mathbf{u}^o(\boldsymbol{\varepsilon}) + \tfrac{\partial}{\partial \boldsymbol{\varepsilon}} \mathbf{h}[\mathbf{x}^o(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}]' \mathbf{w}^o(\boldsymbol{\varepsilon})$.

Jittorntrum (1984) also shows that the local optimal value function f_ℓ^o is twice directionally differentiable under the conditions of Theorem 7.3.

E.2. Sensitivity analysis results for (multi-)parametric quadratic programming problems

This section collects results related to the so-called parametric quadratic programming problems. The literature on parametric programming problems seeks to “fully” characterize the solutions to parameterized programming problems as functions of their parameters. While sensitivity analysis theory addresses the same question in a neighborhood of some parameter values, parametric programming theory seeks to explore the whole parameter space. Of course, this requires considering specific problems such as quadratic programming problems.

The seminal work Bemporal *et al* (2002) lies at the root of a series of articles on parametric quadratic programming in the automatic control literature. It provides elements for the characterization of the solutions to quadratic programming problems on their whole parameter space. It also shows how this characterization can be used for designing efficient algorithms for finite and infinite horizon linear quadratic optimal control problems subject to state and input constraints. This section collects some of the results published in this literature which can be used for solving dynamic programming problems.

It also presents sensitivity analysis results due to Berkelaar *et al* (1997) on specific on parametric quadratic programming problems. These authors address different questions, but their results can be used for complementing those obtained in the automatic control literature.

E.2.1. Parametric (*strictly convex*) *quadratic problems*

Let consider the following quadratic programming problem:

$$\text{QP}(\boldsymbol{\theta}):\ \ \min_{\mathbf{x}\in\mathcal{F}(\boldsymbol{\theta})}\{\mathbf{x}'(\mathbf{h}+\mathbf{F}\boldsymbol{\theta})+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}+1/2\times\boldsymbol{\theta}'\boldsymbol{\Psi}\boldsymbol{\theta}\}$$

where the feasible set $\mathcal{F}(\boldsymbol{\theta})$ of problem QP($\boldsymbol{\theta}$) is the following polyhedron:

$$\mathcal{F}(\boldsymbol{\theta})\equiv\{\mathbf{x}\in\mathbb{R}^N:\mathbf{a}_c'\mathbf{x}=b_c+\mathbf{s}_c'\boldsymbol{\theta}\text{ for }c\in\mathcal{E}\text{ and }\mathbf{a}_c'\mathbf{x}\le b_c+\mathbf{s}_c'\boldsymbol{\theta}\text{ for }c\in\mathcal{J}\}\text{ with }\boldsymbol{\theta}\in\Theta$$

The parameter space $\Theta\subseteq\mathbb{R}^P$ is defined as a polyhedron. Let assume that $\mathcal{E}\cap\mathcal{J}=\varnothing$ and let $\mathcal{C}\equiv\mathcal{E}\cup\mathcal{J}\equiv\{1,...,C\}$ denote the whole set of (non redundant) constraints. Let further assume that $\mathbf{A}\equiv[\mathbf{a}_c':c\in\mathcal{C}]\in\mathbb{R}^{C\times N}$, $\mathbf{b}\equiv[b_c:c\in\mathcal{C}]\in\mathbb{R}^C$ and $\mathbf{S}\equiv[\mathbf{s}_c':c\in\mathcal{C}]\in\mathbb{R}^{C\times P}$, and that the parameter set is full dimensional.

Assuming that **A** is positive semi-definite and that $\mathcal{F}(\boldsymbol{\theta})$ is not empty, the solution in $\mathbf{x} \in \mathcal{F}(\boldsymbol{\theta})$ to the KKT conditions:

$$\begin{cases} \mathbf{h} + \mathbf{F}\boldsymbol{\theta} + \mathbf{H}\mathbf{x} + \mathbf{A}'\boldsymbol{\lambda} = \mathbf{0}, \quad \boldsymbol{\lambda} \in \mathbb{R}^{C} \\ \lambda_c \times (\mathbf{a}_c'\mathbf{x} - b_c - \mathbf{s}_c'\boldsymbol{\theta}) = 0, \quad c \in \mathcal{I} \\ \mathbf{a}_c'\mathbf{x} - b_c - \mathbf{s}_c'\boldsymbol{\theta} = 0, \quad c \in \mathcal{E} \\ \mathbf{a}_c'\mathbf{x} - b_c - \mathbf{s}_c'\boldsymbol{\theta} \le 0, \quad c \in \mathcal{I} \\ \lambda_c \ge 0, \quad c \in \mathcal{I} \end{cases}$$

characterize $\mathcal{X}^*(\boldsymbol{\theta})$. If **H** is positive definite then problem QP($\boldsymbol{\theta}$) is strictly convex in **x** and, as a result, $\mathcal{X}^*(\boldsymbol{\theta})$ reduces to the singleton formed by the unique solution in **x**, $\mathbf{x}^*(\boldsymbol{\theta})$, to problem QP($\boldsymbol{\theta}$), *i.e.* $\mathcal{X}^*(\boldsymbol{\theta}) = \{\mathbf{x}^*(\boldsymbol{\theta})\}$.

The inactive set is defined as:

$$\mathcal{N}(\mathbf{x}, \boldsymbol{\theta}) \equiv \mathcal{C} \setminus \mathcal{A}(\mathbf{x}, \boldsymbol{\theta}).$$

The solution set of problem QP($\boldsymbol{\theta}$) is defined as:

$$\mathcal{X}^*(\boldsymbol{\theta}) \equiv \arg\min_{\mathbf{x} \in \mathcal{F}(\boldsymbol{\theta})} \{\mathbf{x}'(\mathbf{h} + \mathbf{F}\boldsymbol{\theta}) + 1/2 \times \mathbf{x}'\mathbf{H}\mathbf{x} + 1/2 \times \boldsymbol{\theta}'\boldsymbol{\Psi}\boldsymbol{\theta}\}$$

and the corresponding optimal active set $\mathcal{A}^*(\boldsymbol{\theta})$ is the set of constraints which are active for all $\mathbf{x} \in \mathcal{X}^*(\boldsymbol{\theta})$:

$$\mathcal{A}^*(\boldsymbol{\theta}) \equiv \{c \in \mathcal{A}(\mathbf{x}, \boldsymbol{\theta}) \text{ for any } \mathbf{x} \in \mathcal{X}^*(\boldsymbol{\theta})\}.$$

For a given active set $\mathcal{A}$, the matrices $\mathbf{A}_{\mathcal{A}}$, $\mathbf{b}_{\mathcal{A}}$ and $\mathbf{S}_{\mathcal{A}}$ are formed by selecting the rows of the matrices **A**, **b** and **S** belonging to $\mathcal{A}$. The Linear Independence Constraint Qualification (LICQ) holds for the active set $\mathcal{A}$ if and only if $\mathbf{A}_{\mathcal{A}}$ has full row rank.

Let $\mathbf{x} \in \mathcal{F}(\boldsymbol{\theta})$, the set of active constraints or active set at **x** is defined as:

$$\mathcal{A}(\mathbf{x}, \boldsymbol{\theta}) \equiv \{c \in \mathcal{C} : \mathbf{a}_c'\mathbf{x} = b_c + \mathbf{s}_c'\boldsymbol{\theta}\}.$$

The inactive set is defined as:

$$\mathcal{N}(\mathbf{x},\boldsymbol{\theta}) \equiv \mathcal{C} \setminus \mathcal{A}(\mathbf{x},\boldsymbol{\theta}).$$

The solution set of problem QP($\boldsymbol{\theta}$) is defined as:

$$\mathcal{X}^*(\boldsymbol{\theta}) \equiv \arg\min_{\mathbf{x}\in\mathcal{F}(\boldsymbol{\theta})}\{\mathbf{x}'(\mathbf{h}+\mathbf{F}\boldsymbol{\theta})+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}+1/2\times\boldsymbol{\theta}'\boldsymbol{\Psi}\boldsymbol{\theta}\}$$

and the corresponding optimal active set $\mathcal{A}^*(\boldsymbol{\theta})$ is the set of constraints which are active for all $\mathbf{x}\in\mathcal{X}^*(\boldsymbol{\theta})$:

$$\mathcal{A}^*(\boldsymbol{\theta}) \equiv \{c\in\mathcal{A}(\mathbf{x},\boldsymbol{\theta}) \text{ for any } \mathbf{x}\in\mathcal{X}^*(\boldsymbol{\theta})\}.$$

Proposition E1. (Bemporad *et al*, 2002). Let consider problem QP($\boldsymbol{\theta}$) and let assume that $\mathbf{H}$ is positive definite and that:

$$\begin{bmatrix}\boldsymbol{\Psi} & \mathbf{F}' \\ \mathbf{F} & \mathbf{H}\end{bmatrix} \text{ is positive semi-definite.}$$

Then:

(*a*) The set of feasible parameters, *i.e.* $\Theta^* \equiv \{\boldsymbol{\theta}\in\Theta : \mathcal{F}(\boldsymbol{\theta})\neq\emptyset\}$, is polyhedral.

(*b*) The value function $V^*(\boldsymbol{\theta}):\Theta^*\to\mathbb{R}$ to problem QP($\boldsymbol{\theta}$) defined by

$$V^*(\boldsymbol{\theta}) \equiv \min_{\mathbf{x}\in\mathcal{F}(\boldsymbol{\theta})}\{\mathbf{x}'(\mathbf{h}+\mathbf{F}\boldsymbol{\theta})+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}+1/2\times\boldsymbol{\theta}'\boldsymbol{\Psi}\boldsymbol{\theta}\}$$

is piecewise quadratic, continuous and convex in $\boldsymbol{\theta}$ on Θ^*,

(*c*) The solution function in $\mathbf{x}$ to problem QP($\boldsymbol{\theta}$) is unique and defines the solution function $\mathbf{x}^*(\boldsymbol{\theta}):\Theta^*\to\mathbb{R}^N$ by :

$$\mathbf{x}^*(\boldsymbol{\theta}) \equiv \min_{\mathbf{x}\in\mathcal{F}(\boldsymbol{\theta})}\{\mathbf{x}'(\mathbf{h}+\mathbf{F}\boldsymbol{\theta})+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}+1/2\times\boldsymbol{\theta}'\boldsymbol{\Psi}\boldsymbol{\theta}\}.$$

$\mathbf{x}^*$ is piecewise affine and continuous in $\boldsymbol{\theta}$ on Θ^*.

(*e*) If the (LICQ) condition holds for $\mathcal{A}^*(\boldsymbol{\theta})$ on Θ^* then the solution in $\boldsymbol{\lambda}$ to the KKT conditions is also unique and defines the solution function $\boldsymbol{\lambda}^*(\boldsymbol{\theta}):\Theta^*\to\mathbb{R}^C$. $\boldsymbol{\lambda}^*$ is piecewise affine and continuous in $\boldsymbol{\theta}$ on Θ^*.

Proof. See Bemporad *et al* (2002) and Tondel *et al* (2003).

To choose an active set $\mathcal{A}$ allows selecting a linear sub-system of equality constraints from the KKT conditions for calculating the optimal values of $(\mathbf{x},\boldsymbol{\lambda})$. If $\mathcal{A}$ is an optimal active set for $\boldsymbol{\theta}\in\Theta^*$ then we have:

$$\begin{bmatrix}\mathbf{H} & \mathbf{A}'_{\mathcal{A}}\\ \mathbf{A}_{\mathcal{A}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}^*(\boldsymbol{\theta})\\ \boldsymbol{\lambda}^*(\boldsymbol{\theta})\end{bmatrix}=\begin{bmatrix}-\mathbf{h}\\ \mathbf{b}_{\mathcal{A}}\end{bmatrix}+\begin{bmatrix}-\mathbf{F}\\ \mathbf{S}_{\mathcal{A}}\end{bmatrix}\boldsymbol{\theta}.$$

This equation system can be solved by the standard null space method.

Proposition E2. *Optimal solutions and critical regions*.

Let consider problem QP($\boldsymbol{\theta}$) and an arbitrary active set $\mathcal{A}$. Let $M_{\mathcal{A}}$ denote the cardinality of $\mathcal{A}$ and $\mathcal{N}_{\mathcal{A}}\equiv\mathcal{C}\setminus\mathcal{A}$. Let assume that $\mathcal{A}$ satisfies the LICQ condition and that $\mathbf{H}$ is positive definite. Let $\mathbf{Z}_{\mathcal{A}}\in\mathbb{R}^{N\times(N-M_{\mathcal{A}})}$ be a matrix whose column spans the null space of $\mathbf{A}_{\mathcal{A}}$ and let $\mathbf{Y}_{\mathcal{A}}\in\mathbb{R}^{N\times M_{\mathcal{A}}}$ be a matrix such that $[\mathbf{Z}_{\mathcal{A}}\ \ \mathbf{Y}_{\mathcal{A}}]$ is non-singular.

Then, for any $\boldsymbol{\theta}\in\Theta^*$ such that $\mathcal{A}$ is an optimal active set:

(*a*) the optimal solutions in **s** and in $\boldsymbol{\lambda}$ to the KKT conditions are unique and defined by:

$$\begin{cases}\mathbf{x}^*_{\mathcal{A}}(\boldsymbol{\theta})=\mathbf{K}^x_{\mathcal{A}}\boldsymbol{\theta}+\boldsymbol{\kappa}^x_{\mathcal{A}}\\ \boldsymbol{\lambda}^*_{\mathcal{A}}(\boldsymbol{\theta})=\mathbf{K}^\lambda_{\mathcal{A}}\boldsymbol{\theta}+\boldsymbol{\kappa}^\lambda_{\mathcal{A}}\end{cases}$$

where:

$$\begin{cases}\mathbf{K}^x_{\mathcal{A}}\equiv\mathbf{Y}_{\mathcal{A}}(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}\mathbf{S}_{\mathcal{A}}-\mathbf{Z}_{\mathcal{A}}(\mathbf{Z}'_{\mathcal{A}}\mathbf{H}\mathbf{Z}_{\mathcal{A}})^{-1}\mathbf{Z}'_{\mathcal{A}}\left(\mathbf{F}+\mathbf{H}\mathbf{Y}_{\mathcal{A}}(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}\mathbf{S}_{\mathcal{A}}\right)\\ \boldsymbol{\kappa}^x_{\mathcal{A}}\equiv\left(\mathbf{Y}_{\mathcal{A}}-\mathbf{Z}_{\mathcal{A}}(\mathbf{Z}'_{\mathcal{A}}\mathbf{H}\mathbf{Z}_{\mathcal{A}})^{-1}\mathbf{Z}'_{\mathcal{A}}\mathbf{H}\mathbf{Y}_{\mathcal{A}}\right)(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}\mathbf{b}_{\mathcal{A}}-\mathbf{Z}_{\mathcal{A}}(\mathbf{Z}'_{\mathcal{A}}\mathbf{H}\mathbf{Z}_{\mathcal{A}})^{-1}\mathbf{Z}'_{\mathcal{A}}\mathbf{h}\\ \mathbf{K}^\lambda_{\mathcal{A}}\equiv-(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}\mathbf{Y}'_{\mathcal{A}}(\mathbf{H}\mathbf{K}^x_{\mathcal{A}}+\mathbf{F})\\ \boldsymbol{\kappa}^\lambda_{\mathcal{A}}\equiv-(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}\mathbf{Y}'_{\mathcal{A}}(\mathbf{H}\boldsymbol{\kappa}^x_{\mathcal{A}}+\mathbf{h})\end{cases}$$

(*b*) the active set $\mathcal{A}$ is the unique optimal active set in the interior in the critical region defined by:

$$CR_{\mathcal{A}} \equiv \{\boldsymbol{\theta} \in \Theta^* : \mathbf{G}_{\mathcal{A}}\boldsymbol{\theta} \leq \mathbf{g}_{\mathcal{A}}\}$$

where:

$$\mathbf{G} \equiv \begin{bmatrix} \mathbf{A}_{\mathcal{N}_{\mathcal{A}}}\mathbf{K}^{x}_{\mathcal{A}} - \mathbf{S}_{\mathcal{N}_{\mathcal{A}}} \\ (\mathbf{K}^{\lambda}_{\mathcal{A}})_{\mathcal{J}\cap\mathcal{A}} \end{bmatrix} \text{ and } \mathbf{g}_{\mathcal{A}} \equiv \begin{bmatrix} \mathbf{b}_{\mathcal{N}_{\mathcal{A}}} - \mathbf{A}_{\mathcal{N}_{\mathcal{A}}}\boldsymbol{\kappa}^{x}_{\mathcal{A}} \\ (\boldsymbol{\kappa}^{\lambda}_{\mathcal{A}})_{\mathcal{J}\cap\mathcal{A}} \end{bmatrix}.$$

Proof. See Tondel *et al* (2003). Note that these authors do not assume that **H** is positive definite but rather assume that $\mathbf{Z}'_{\mathcal{A}}\mathbf{H}\mathbf{Z}_{\mathcal{A}} > 0$.

This proposition has two main implications. First, to know the optimal active set associated to any $\boldsymbol{\theta} \in \Theta^*$, *i.e.* $\mathcal{A}^*(\boldsymbol{\theta})$, allows easily computing the solutions to problem QP($\boldsymbol{\theta}$). It suffices to use the formulas for $\mathbf{x}^*_{\mathcal{A}^*(\boldsymbol{\theta})}(\boldsymbol{\theta})$ and $\boldsymbol{\lambda}^*_{\mathcal{A}^*(\boldsymbol{\theta})}(\boldsymbol{\theta})$. Second, there exists a unique polyhedral partition of the feasible parameter space Θ^* such that the interior of each polyhedral subset of Θ^* contains the parameters $\boldsymbol{\theta}$ for which a given active set is the unique optimal active set. This polyhedral partition of Θ^* offers a full characterization of the optimal active sets associated to the solutions to problem QP($\boldsymbol{\theta}$). Once this polyhedral partition of Θ^* is known, to obtain the solutions problem QP($\boldsymbol{\theta}$) just requires (*a*) simple evaluations aimed at identifying the unique optimal active set of $\boldsymbol{\theta}$ and then (*b*) to apply the corresponding formulas to obtain $\mathbf{x}^*(\boldsymbol{\theta})$ and $\boldsymbol{\lambda}^*(\boldsymbol{\theta})$, and thus $V^*(\boldsymbol{\theta})$. Bemporad et al (2002), Tondel *et al* (2003) or Gupta *et al* (2011) propose algorithms aimed at characterizing the relevant partition of Θ^*.

The matrices $\mathbf{Z}_{\mathcal{A}}$ and $\mathbf{Y}_{\mathcal{A}}$ can be chosen such that $(\mathbf{A}_{\mathcal{A}}\mathbf{Y}_{\mathcal{A}})^{-1}$ is easily obtained. This can be done by choosing a QR characterization of $\mathbf{A}'_{\mathcal{A}}$, *i.e.*:

$$\mathbf{A}'_{\mathcal{A}}\mathbf{P} = \begin{bmatrix} \mathbf{Q}_{1,\mathcal{A}} & \mathbf{Q}_{2,\mathcal{A}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_{1,\mathcal{A}}\mathbf{R} \quad \text{with} \quad \begin{bmatrix} \mathbf{Y}_{\mathcal{A}} & \mathbf{Z}_{\mathcal{A}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{Q}_{1,\mathcal{A}} & \mathbf{Q}_{2,\mathcal{A}} \end{bmatrix},$$

where **P** is a permutation matrix (and where $\mathbf{Q}_{2,\mathcal{A}}$ and **0** may be “empty”).

If the (LICQ) condition doesn’t hold for some $\mathcal{A}^*(\boldsymbol{\theta})$ then the solution in $\boldsymbol{\lambda}$ to the KKT conditions is not unique and several optimal combinations of active constraints exist. This difficulty can be overcome either by suitably removing redundant constraints or use projections in the $(\boldsymbol{\theta},\boldsymbol{\lambda})$-space. In this last solution option, the set $\boldsymbol{\lambda}^*(\boldsymbol{\theta})$ can be characterized as a polyhedron in the $(\boldsymbol{\theta},\boldsymbol{\lambda})$-space.

Proposition E3. *Multiple solutions in* $\boldsymbol{\lambda}$.

Let consider problem QP($\boldsymbol{\theta}$) and an arbitrary active set $\mathcal{A}$. Let $M_{\mathcal{A}}$ denote the cardinality of $\mathcal{A}$ and $\mathcal{N}_{\mathcal{A}} \equiv \mathcal{C} \setminus \mathcal{A}$. Let assume **H** is positive definite and that $\mathcal{A}$ doesn’t satisfy the LICQ condition. Let $\mathbf{Z}^{\mathcal{C}}_{\mathcal{A}}$ be a matrix whose column spans the null space of $\mathbf{A}'_{\mathcal{A}}$ and let $\mathbf{Y}^{\mathcal{C}}_{\mathcal{A}}$ be a matrix such that $[\mathbf{Z}^{\mathcal{C}}_{\mathcal{A}} \ \mathbf{Y}^{\mathcal{C}}_{\mathcal{A}}]$ is non-singular.

Then, for any $\boldsymbol{\theta} \in \Theta^*$ such that $\mathcal{A}$ is an optimal active set:

(*a*) The optimal solutions in $\boldsymbol{\lambda}$ to the KKT conditions are charcterized by:

$$\boldsymbol{\lambda}^*_{\mathcal{A}}(\boldsymbol{\theta}) = \mathbf{K}^{\mathcal{C},\lambda}_{\mathcal{A}}\boldsymbol{\theta} + \boldsymbol{\kappa}^{\mathcal{C},\lambda}_{\mathcal{A}} + \mathbf{Z}^{\mathcal{C}}_{\mathcal{A}}\boldsymbol{\mu}$$

where:

$$\begin{cases} \mathbf{K}^{\mathcal{C},\lambda}_{\mathcal{A}} \equiv -\mathbf{Y}^{\mathcal{C}}_{\mathcal{A}}(\mathbf{Y}'_{\mathcal{A}}\mathbf{A}'_{\mathcal{A}}\mathbf{Y}^{\mathcal{C}}_{\mathcal{A}})^{-1}\mathbf{Y}'_{\mathcal{A}}(\mathbf{H}\mathbf{K}^{x}_{\mathcal{A}} + \mathbf{F}) \\ \boldsymbol{\kappa}^{\mathcal{C},\lambda}_{\mathcal{A}} \equiv -\mathbf{Y}^{\mathcal{C}}_{\mathcal{A}}(\mathbf{Y}'_{\mathcal{A}}\mathbf{A}'_{\mathcal{A}}\mathbf{Y}^{\mathcal{C}}_{\mathcal{A}})^{-1}\mathbf{Y}'_{\mathcal{A}}(\mathbf{H}\boldsymbol{\kappa}^{x}_{\mathcal{A}} + \mathbf{h}) \end{cases}$$

and where $\boldsymbol{\mu}$ is any vector such that:

$$(\mathbf{K}_{\mathcal{A}}^{C,\lambda})_{\mathcal{I}\cap\mathcal{A}}\boldsymbol{\theta}+(\boldsymbol{\kappa}_{\mathcal{A}}^{C,\lambda})_{\mathcal{I}\cap\mathcal{A}}+(\mathbf{Z}_{\mathcal{A}}^{C})_{\mathcal{I}\cap\mathcal{A}}\boldsymbol{\mu}\geq\mathbf{0}.$$

(b) The active set $\mathcal{A}$ is optimal in the interior of the projection of

$$\begin{cases}\mathbf{A}_{\mathcal{N}_{\mathcal{A}}}\mathbf{x}_{\mathcal{A}}^{*}(\boldsymbol{\theta})-\mathbf{b}_{\mathcal{N}_{\mathcal{A}}}-\mathbf{S}_{\mathcal{N}_{\mathcal{A}}}\boldsymbol{\theta}\leq\mathbf{0}\\(\mathbf{K}_{\mathcal{A}}^{C,\lambda})_{\mathcal{I}\cap\mathcal{A}}\boldsymbol{\theta}+(\boldsymbol{\kappa}_{\mathcal{A}}^{C,\lambda})_{\mathcal{I}\cap\mathcal{A}}+(\mathbf{Z}_{\mathcal{A}}^{C})_{\mathcal{I}\cap\mathcal{A}}\boldsymbol{\mu}\geq\mathbf{0}\end{cases}$$

onto the $\boldsymbol{\theta}$-space.

Proof. See Tondel *et al* (2003).

E.2.2. Sensitivity analysis results for a class of parametric quadratic problems

Berkelaar *et al* (1997) consider (among other problems) the following parametric quadratic programming problem:

$$\mathrm{P}_{(\theta,\vartheta)}:\ \ \min_{\mathbf{x}\in\mathcal{F}(\vartheta)}\{\mathbf{x}'(\mathbf{h}+\theta\times\boldsymbol{\delta}_h)+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\}$$

where:

$$\mathcal{F}(\vartheta)\equiv\{\mathbf{x}\in\mathbb{R}^N:\mathbf{A}\mathbf{x}=\mathbf{b}+\vartheta\times\boldsymbol{\delta}_h\text{ and }\mathbf{x}\geq\mathbf{0}\}.$$

and:

$$\mathbf{A}\mathbf{x}=\mathbf{b}+\vartheta\times\boldsymbol{\delta}_b\quad\Leftrightarrow\quad\mathbf{a}_c'\mathbf{x}=b_c+\vartheta\delta_{b,c}\text{ for }c\in C.$$

Their theorems 50 and 58 provide the left- and right-derivatives in (θ,ϑ) of the value function of problem $\mathrm{P}_{(\theta,\vartheta)}$:

$$W^o(\theta,\vartheta)\equiv\min_{\mathbf{x}\in\mathcal{F}(\vartheta)}\{\mathbf{x}'(\mathbf{h}+\theta\times\boldsymbol{\delta}_h)+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\}$$

assuming that the feasible set $\mathcal{F}(\theta)$ has a non-empty interior and that $\mathbf{H}$ is positive semidefinite.

The next proposition collects the results of theorems 50 and 58 in Berkelaar *et al* (1997).

Proposition E4. *Value function directional derivatives.*

Let consider problem $\mathrm{P}_{(\theta,\vartheta)}$ and let assume that $\mathcal{F}(\theta)$ has a non-empty interior and that **H** is positive semidefinite. Let $\mathrm{D}_{(\theta,\vartheta)}$ denote the Wolfe-dual problem associated to problem $\mathrm{P}_{(\theta,\vartheta)}$:

$$\mathrm{D}_{(\theta,\vartheta)}: \ \max_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{G}(\theta,\mathbf{c})}\{\boldsymbol{\lambda}'(\mathbf{b}+\vartheta\times\boldsymbol{\delta}_b)-1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\}$$

where:

$$\mathcal{G}(\theta)\equiv\{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathbb{R}^{2N}\times\mathbb{R}^{C}:-\mathbf{H}\mathbf{x}+\boldsymbol{\lambda}+\mathbf{A}'\boldsymbol{\mu}=\mathbf{h}+\theta\times\boldsymbol{\delta}_h \text{ and } \boldsymbol{\lambda}\geq\mathbf{0}\}.$$

and let $\mathcal{L}^o(\theta,\vartheta)$ be the solution set in $(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ to $\mathrm{D}_{(\theta,\vartheta)}$. Let $\mathcal{S}_{(\theta,\vartheta)}$ denote the (possibly empty) subset of $\mathcal{N}\equiv\{1,...,N\}$ defined by:

$$\mathcal{S}_{(\theta,\vartheta)}\equiv\{n\in\mathcal{N}:\exists(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(\theta,\vartheta):\lambda_n=x_n=0\}.$$

(*a*) The KKT conditions characterizing the solutions in $(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ to problem $\mathrm{D}_{(\theta,\vartheta)}$ are:

$$\begin{cases}(\mathbf{h}+\theta\times\boldsymbol{\delta}_h)+\mathbf{H}\mathbf{x}-\mathbf{A}'\boldsymbol{\mu}-\boldsymbol{\lambda}=\mathbf{0}\\ \mathbf{A}\mathbf{x}-(\mathbf{b}+\vartheta\times\boldsymbol{\delta}_b)=\mathbf{0}\\ \boldsymbol{\lambda}\geq\mathbf{0},\ \mathbf{x}\geq\mathbf{0},\ \mathbf{x}'\boldsymbol{\lambda}=0\end{cases}$$

(*b*) Let $(\mathbf{x}^o,\boldsymbol{\lambda}^o,\boldsymbol{\mu}^o)\in\mathcal{L}^o(\theta,0)$, then the derivatives of $W^o(\theta,0)$ in θ at $(\theta,0)$ satisfy:

$$\tfrac{\partial}{\partial\theta^-}W^o(\theta,0)=\min_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(\theta,0)}\{\boldsymbol{\delta}_h'\mathbf{x}\ \text{ s.t. }\ \boldsymbol{\lambda}'\mathbf{x}^o=\mathbf{x}'\boldsymbol{\lambda}^o=0,\ x_n=\lambda_n=0 \text{ for } n\in\mathcal{S}_{(\theta,0)}\}$$

and:

$$\tfrac{\partial}{\partial\theta^+}W^o(\theta,0)=\max_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(\theta,0)}\{\boldsymbol{\delta}_h'\mathbf{x}\ \text{ s.t. }\ \boldsymbol{\lambda}'\mathbf{x}^o=\mathbf{x}'\boldsymbol{\lambda}^o=0,\ x_n=\lambda_n=0 \text{ for } n\in\mathcal{S}_{(\theta,0)}\}.$$

(*c*) Let $(\mathbf{x}^o,\boldsymbol{\lambda}^o,\boldsymbol{\mu}^o)\in\mathcal{L}^o(0,\vartheta)$, then the derivatives of $W^o(0,\vartheta)$ in ϑ at $(0,\vartheta)$ satisfy:

$$\tfrac{\partial}{\partial\theta^-}W^o(0,\vartheta)=\min_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(0,\vartheta)}\{\boldsymbol{\delta}_b'\boldsymbol{\mu}\ \text{ s.t. }\ \boldsymbol{\lambda}'\mathbf{x}^o=\mathbf{x}'\boldsymbol{\lambda}^o=0,\ x_n=\lambda_n=0 \text{ for } n\in\mathcal{S}_{(0,\vartheta)}\}$$

and:

$$\tfrac{\partial}{\partial\theta^+}W^o(0,\vartheta)=\max_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(0,\vartheta)}\{\boldsymbol{\delta}_b'\boldsymbol{\mu}\ \text{ s.t. }\ \boldsymbol{\lambda}'\mathbf{x}^o=\mathbf{x}'\boldsymbol{\lambda}^o=0,\ x_n=\lambda_n=0 \text{ for } n\in\mathcal{S}_{(0,\vartheta)}\}.$$

Proof. See Berkelaar *et al* (1997), theorems 50 and 58.

The next proposition is a corollary to Proposition E4. The directional derivative results given in this last proposition provide the left- and right- partial derivatives of $W^o(0,0)$ in the elements of $(\mathbf{h},\mathbf{b})$.

Let now consider the parametric quadratic programming problem:

$$\mathrm{P}_{(\mathbf{h},\mathbf{b})}:\ \ \min_{\mathbf{x}}\{\mathbf{x}'\mathbf{h}+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\text{ s.t. }\mathbf{x}\geq\mathbf{0}\text{ and }\mathbf{A}\mathbf{x}=\mathbf{b}\},$$

its value function:

$$V^o(\mathbf{h},\mathbf{b})\equiv\min_{\mathbf{x}}\{\mathbf{x}'\mathbf{h}+1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\text{ s.t. }\mathbf{x}\geq\mathbf{0}\text{ and }\mathbf{A}\mathbf{x}=\mathbf{b}\}$$

and its associated Wolfe-dual problem:

$$\mathrm{D}_{(\mathbf{h},\mathbf{b})}:\ \ \max_{(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})}\{\boldsymbol{\lambda}'\mathbf{b}-1/2\times\mathbf{x}'\mathbf{H}\mathbf{x}\text{ s.t. }\boldsymbol{\lambda}\geq\mathbf{0}\text{ and }-\mathbf{H}\mathbf{x}+\boldsymbol{\lambda}+\mathbf{A}'\boldsymbol{\mu}=\mathbf{h}\}.$$

Proposition E5. *Value function partial derivatives*.

Let consider problem $\mathrm{P}_{(\mathbf{h},\mathbf{b})}$ and its Wolfe-dual problem $\mathrm{D}_{(\mathbf{h},\mathbf{b})}$. Let assume that the feasible set $\mathcal{F}(\mathbf{h},\mathbf{b})\equiv\{\mathbf{x}\in\mathbb{R}^N:\mathbf{x}\geq\mathbf{0}\text{ and }\mathbf{A}\mathbf{x}=\mathbf{b}\}$ has a non-empty interior and that $\mathbf{H}$ is positive definite. Let $\mathcal{L}^o(\mathbf{h},\mathbf{b})\subset\mathbb{R}_+^{2N}\times\mathbb{R}^C$ denote the solution set in $(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ to problem $\mathrm{D}_{(\mathbf{h},\mathbf{b})}$. Let $\mathcal{S}_{(\theta,\vartheta)}$ denote the (possibly empty) subset of $\mathcal{N}\equiv\{1,...,N\}$ defined by:

$$\mathcal{S}_{(\theta,\vartheta)}\equiv\{n\in\mathcal{N}:\exists(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{L}^o(\mathbf{h},\mathbf{b}):\lambda_n=x_n=0\}.$$

(a) The KKT conditions characterizing the solutions in $(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ to problem $\mathrm{D}_{(\mathbf{h},\mathbf{b})}$ are:

$$\begin{cases}\mathbf{h}+\mathbf{H}\mathbf{x}-\mathbf{A}'\boldsymbol{\mu}-\boldsymbol{\lambda}=\mathbf{0}\\ \mathbf{A}\mathbf{x}-\mathbf{b}=\mathbf{0}\\ \boldsymbol{\lambda}\geq\mathbf{0},\ \mathbf{x}\geq\mathbf{0},\ \mathbf{x}'\boldsymbol{\lambda}=0\end{cases}$$

(b) The solution in $\mathbf{x}$ to problems $\mathrm{P}_{(\mathbf{h},\mathbf{b})}$ and $\mathrm{D}_{(\mathbf{h},\mathbf{b})}$ is unique. Let $\mathbf{x}^o$ denote this solution.

$\mathcal{L}^o(\mathbf{h},\mathbf{b})$ can be defined as $\{\mathbf{x}^o\}\times\mathcal{M}^o(\mathbf{h},\mathbf{b})$ where $\mathcal{M}^o(\mathbf{h},\mathbf{b})\subset\mathbb{R}_+^N\times\mathbb{R}^C$.

(c) Let $(\boldsymbol{\lambda}^o,\boldsymbol{\mu}^o)\in\mathcal{M}^o(\mathbf{h},\mathbf{b})$, then the derivatives of $V^o(\mathbf{h},\mathbf{b})$ in h_n at $(\mathbf{h},\mathbf{b})$ satisfy:

$$\tfrac{\partial}{\partial h_n^-}V^o(\mathbf{h},\mathbf{b})=\tfrac{\partial}{\partial h_n^+}V^o(\mathbf{h},\mathbf{b})=x_n^o=\tfrac{\partial}{\partial h_n}V^o(\mathbf{h},\mathbf{b}).$$

(d) Let $(\boldsymbol{\lambda}^o,\boldsymbol{\mu}^o)\in\mathcal{M}^o(\mathbf{h},\mathbf{b})$, then the derivatives of $V^o(\mathbf{h},\mathbf{b})$ in b_c at $(\mathbf{h},\mathbf{b})$ satisfy:

$$\tfrac{\partial}{\partial b_c^-}V^o(\mathbf{h},\mathbf{b})=\min_{(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{M}^o(\mathbf{h},\mathbf{b})}\{\mu_c \text{ s.t. } \boldsymbol{\lambda}'\mathbf{x}^o=0 \text{ and } \lambda_n=0 \text{ for } n\in\mathcal{S}_{(\mathbf{h},\mathbf{b})}\}$$

and:

$$\tfrac{\partial}{\partial b_c^+}V^o(\mathbf{h},\mathbf{b})=\max_{(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathcal{M}^o(\mathbf{h},\mathbf{b})}\{\mu_c \text{ s.t. } \boldsymbol{\lambda}'\mathbf{x}^o=0 \text{ and } \lambda_n=0 \text{ for } n\in\mathcal{S}_{(\mathbf{h},\mathbf{b})}\}.$$

Proof. Problem $\mathrm{P}_{(\mathbf{h},\mathbf{b})}$ is strictly convex since it involves linear constraints only and **H** is definite positive. It is also strongly dual because its feasible set $\mathcal{F}(\mathbf{h},\mathbf{b})$ has a non-empty interior. These properties of problem $\mathrm{P}_{(\mathbf{h},\mathbf{b})}$ ensure results (a) and (b). Results (b)-(d) are direct applications of Proposition E4.

QED.

If the solution in μ_c to problem $\mathrm{D}_{(\mathbf{h},\mathbf{b})}$ is unique then:

$$\tfrac{\partial}{\partial b_c^-}V^o(\mathbf{h},\mathbf{b})=\tfrac{\partial}{\partial b_c^+}V^o(\mathbf{h},\mathbf{b})=\mu_c^o=\tfrac{\partial}{\partial b_c}V^o(\mathbf{h},\mathbf{b}).$$

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